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# Galilei-invariant equations for massive fields

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**Abstract**

Galilei-invariant equations for massive fields with various spins have been found and classified. They have been derived directly, i.e., by using requirement of the Galilei invariance and various facts on representations of the Galilei group deduced in the paper written by de Montigny, Niederle and Nikitin (2006 *J. Phys. A: Math. Gen.* **39** 1–21). A completed list of non-equivalent Galilei-invariant wave equations for vector and scalar fields is presented. It shows two things. First that the collection of such equations is very broad and describes many physically consistent systems. In particular it is possible to describe spin–orbit and Darwin couplings in frames of a Galilei-invariant approach. Second, these Galilei-invariant equations can be obtained either via contraction of known relativistic equations or via contractions of quite new relativistic wave equations.

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## 1. Introduction

It is well known that the Galilei group $G(1,3)$ and its representations play the same role in non-relativistic physics as the Poincaré group $P(1,3)$ and its representations do in the relativistic case. In fact the Galilei group and its representations form a group-theoretical basis of classical mechanics and electrodynamics. They replace the Poincaré group and its representations whenever velocities of bodies are much smaller than the speed of light in vacuum. On the other hand, structures of subgroups of the Galilei group and of its representations are in many respects more complex than those of the Poincaré group and therefore it is not perhaps so surprising that representations of the Poincaré group were described in [1] almost 15 years earlier than representations of the Galilei group [2] in spite of the fact that the relativity principle of classical physics was formulated by Galilei in 1632, i.e., about three centuries prior to that of relativistic physics by Einstein.

An excellent review of representations of the Galilei group was written by Lévy-Leblond [3]. It appears that the Galilei group, as distinct from the Poincaré group, has besides ordinary representations also projective ones (see [2] and [4] respectively). However its subgroup—the homogeneous Galilei group $HG(1, 3)$ which plays in non-relativistic physics the role of the Lorentz group in the relativistic case—has a more complex structure so that its finite-dimensional indecomposable representations are not classifiable (for details see [5]). And they are the representations which play a key role in the description of physical systems satisfying the Galilei relativity principle!

An important class of indecomposable finite-dimensional representations of the group $HG(1, 3)$ was found and completely classified in [5]. It contains all representations of the homogeneous Galilei group which, when restricted to its rotation subgroup, decompose to spins 0, $1/2$ and 1 representations. It was explained in [5] and [6] how these representations can be obtained from those of the Lorentz group by means of the Inönü–Wigner contractions.

The representations classified in [5] are interesting from both physical and mathematical points of view. Physically, the related spin values exhaust all ones observed experimentally for stable particles. Mathematically, the results of paper [5] are unimprovable in the sense that the problem of classification of the representations with other spin content is unsolvable in general. This gives us undreamt possibilities of describing various non-relativistic (quantum-mechanical and field-theoretical) systems of interacting particles and fields with spins 0, $1/2$ and 1. For instance, it was possible to find the most general Pauli interaction of the Galilean spin-$1/2$ particles with an external electromagnetic field, see [5].

Starting with the indecomposable representations of the group $HG(1, 3)$ found in [5] an extended class of linear and nonlinear equations for Galilean massless fields had been found in our previous paper [7]. In particular, Galilean analogues of the Born–Infeld and Maxwell–Chern–Simon systems were discussed therein. Moreover, in fact all possible Galilei-invariant systems of first-order equations for massless vector and scalar fields had been classified in [7].

In the present paper we continue and complete the research started in [5–7]. Namely, we study first vector and spinor representations of the homogeneous Galilei group in detail and then using them we construct various wave equations for *massive* particles with spins 0, $1/2$, 1 and $3/2$. In this way a completed list of systems of Galilei invariant first-order partial differential equations is found which describe particles with spin $s < 3/2$.

There are well-developed methods for deriving relativistic wave equations invariant w.r.t. the Poincaré group[3]. Consequently several systematic approaches to relativistic equations are now available, the most popular of which are those by Bhabha and Gelfand and Yaglom [9], those by Bruhat [10] and those based on Gårding's technique [11]. These approaches can be used for construction of the Galilei-invariant equations as well.

First, we begin with the Bhabha and Gelfand–Yaglom [9] approach which is a direct extension of the method yielding the Dirac equation. The corresponding relativistic wave equations can be written as systems of the first-order partial-differential linear equations of the form

$$(\beta_\mu p^\mu + \beta_4 m)\Psi(\mathbf{x}, t) = 0, \tag{1}$$

where $p^0 = \mathrm{i}\frac{\partial}{\partial x_0}$, $p^a = \mathrm{i}\frac{\partial}{\partial x_a}$ $(a = 1, 2, 3)$, and $\beta_\mu$ $(\mu = 0, 1, 2, 3)$ and $\beta_4$ are square matrices restricted by the condition of Poincaré invariance. Note that in the relativistic approach matrix $\beta_4$ is usually assumed to be proportional to a unit matrix.

[3] These methods have had a long and tangled history. It began with Schrödinger's proposal of a relativistic equation in 1926, Dirac's equation for massive spin $1/2$ particle in 1928 and Majorana's pioneering study in 1932 and has always been a subject of interest of many scientists (Bhabha, Harish-Chandra, Wild, Fierz, Pauli, Kemmer, Duffin, Bargmann, Wigner, Weinberg, Schwinger, and so on, see, e.g., [8] and references cited therein).

The theory of the Poincaré-invariant equations (1) is clearly explained in detail for instance in the Gel'fand–Minlos–Shapiro book [12].

The other approaches make use of tensor calculus, and the associated equations have the form of covariant vectors or tensors (see, e.g., [13]). A popular example is the first-order Proca equation [14],

$$p^\mu \Psi^\nu - p^\nu \Psi^\mu = \kappa \Psi^{\mu\nu}, \qquad p_\nu \Psi^{\nu\mu} = \kappa \Psi^\mu, \tag{2}$$

where $\Psi^\mu$ and $\Psi^{\mu\nu}$ are a 4-vector and a skew-symmetric spinor respectively which transform according to the representation $D\left(\frac{1}{2}, \frac{1}{2}\right) \oplus D(1, 0) \oplus D(0, 1)$ of the Lorentz group. The second-order Proca [14], Rarita–Schwinger [15] and Singh–Hagen [16] equations serve as other examples. Let us mention that all these relativistic equations violate causality or predict incorrect values for the gyromagnetic ratio $g$. The tensor-spinorial equations for particles with an arbitrary half-integer spin which are not violating causality and admit the right value for $g$ were discussed in detail in [17].

In the present paper we use both the above-mentioned approaches and derive the Galilei-invariant equations for particles with spins 0, $\frac{1}{2}$, 1 and $\frac{3}{2}$. Moreover we present a complete list of the Galilei-invariant equations (1) for scalar and vector fields.

A special class of Galilei-invariant equations for particles with arbitrary spins $s$ was described in the significant paper [19]. The related equations have the minimal number of components and predict the value $g = 1/s$ for the gyromagnetic ratio. Note that the same value of $g$ is predicted by the relativistic wave equations in frames of the minimal interaction principle.

Some particular results associated with the Galilei-invariant equations (1) can also be found in [3, 5, 18–23]. Galilean analogues of the Bargman–Wigner equations are presented in a recent paper [24]. However, these equations became incompatible whenever a minimal interaction with an external e.m. field was introduced [24].

## 2. The Galilei algebra and Galilei-invariant wave equations

### 2.1. Basic definitions

In this section we shall develop a Galilean version of the Bhabha and Gelfand–Yaglom approach [9] and present a complete list of the corresponding Galilei-invariant wave equations. Let us note that some of these equations are already known (see for instance, [3, 21] and [23]).

Equation (1) is said to be *invariant* w.r.t. the Galilei transformations

$$\mathbf{x} \to \mathbf{x}' = \mathbf{R}\mathbf{x} + \mathbf{v}t + \mathbf{b}, \qquad t \to t' = t + a, \tag{3}$$

where $a$, $\mathbf{b}$, $\mathbf{v}$ are real parameters and $\mathbf{R}$ is a rotation matrix, if function $\Psi$ in (1) cotransforms as

$$\Psi(\mathbf{x}, t) \to \Psi'(\mathbf{x}', t') = \mathrm{e}^{\mathrm{i}f(\mathbf{x},t)} T \Psi(\mathbf{x}, t), \tag{4}$$

i.e., according to a particular representation of the Galilei group. Here $T$ is a matrix depending on transformation parameters only, $f(\mathbf{x}, t) = m(\mathbf{v} \cdot \mathbf{x} + tv^2/2 + c)$, $c$ is an arbitrary constant and $\Psi'(\mathbf{x}', t')$ satisfies the same equation in prime variables as $\Psi(\mathbf{x}, t)$ in the initial ones.

The Lie algebra corresponding to representation (4) has the following generators:

$$\begin{aligned}
&P_0 = p_0 = \mathrm{i}\frac{\partial}{\partial t}, \qquad P_a = p_a = -\mathrm{i}\frac{\partial}{\partial x^a}, \qquad M = mI,\\
&J_a = \varepsilon_{abc} x_b p_c + S_a \\
&G_a = t p_a - m x_a + \eta_a,
\end{aligned} \tag{5}$$

where $S_a$ and $\eta_a$ are matrices which satisfy the following commutation relations:

$$\begin{aligned}
&[S_a, S_b] = \mathrm{i}\varepsilon_{abc} S_c, \\
&[\eta_a, S_b] = \mathrm{i}\varepsilon_{abc}\eta_c, \; [\eta_a, \eta_b] = 0
\end{aligned} \tag{6}$$

that is, they form a basis of the homogeneous Galilei algebra $hg(1,3)$.

Equation (1) is invariant with respect to the Galilei transformations (3), (4), if their generators (5) transform solutions of (1) into solutions. This requirement together with the existence of the Galilei-invariant Lagrangian for (1) yields the following conditions on matrices $\beta_\mu$ ($\mu = 0, 1, 2, 3,$ ) and $\beta_4$ [22]:

$$\begin{aligned}
&\eta_a^\dagger\beta_4 - \beta_4\eta_a = -\mathrm{i}\beta_a, \\
&\eta_a^\dagger\beta_b - \beta_b\eta_a = -\mathrm{i}\delta_{ab}\beta_0, \\
&\eta_a^\dagger\beta_0 - \beta_0\eta_a = 0, \qquad a, b = 1, 2, 3.
\end{aligned} \tag{7}$$

Moreover, $\beta_0$ and $\beta_4$ must be scalars w.r.t. rotations, i.e., they have to commute with $S_a$.

Thus the problem of classification of the Galilei-invariant equations (1) is equivalent to finding matrices $S_a, \eta_a, \beta_0, \beta_a$ and $\beta_4$ satisfying relations (6) and (7). Unfortunately, a subproblem of this problem, i.e., a complete classification of non-equivalent finite-dimensional representations of algebra (6), appears to be in general unsolvable (that is a 'wild' algebraic problem). However, for two important particular cases, i.e., for purely spinor and vector–scalar representations, the problem of finding all finite-dimensional indecomposable representations of the algebra $hg(1,3)$ is solvable and was completely solved in [5].

### *2.2. Spinor fields and the corresponding wave equations*

Let $\tilde{s}$ be the highest value of spin which appears when representation of algebra $hg(1,3)$ is reduced to its subalgebra $so(3)$. Then the corresponding representation space of $hg(1,3)$ is said to be the space of fields of spin $\tilde{s}$.

As mentioned in [5] there exist only two non-equivalent indecomposable representations of the algebra $hg(1,3)$ defined on fields of spin 1/2. One of them, $D_1(\frac{1}{2})$, when restricted to the subalgebra $so(3)$ remains irreducible while the other one, $D_2(\frac{1}{2})$, decomposes to two irreducible representations $D(1/2)$ of $so(3)$. The corresponding matrices $S_a$ and $\eta_a$ can be written in the following form:

$$S_a = \tfrac{1}{2}\sigma_a, \qquad \eta_a = \mathbf{0} \qquad \text{for} \quad D_1\left(\tfrac{1}{2}\right) \tag{8}$$

and

$$S_a = \frac{1}{2}\begin{pmatrix} \sigma_a & \mathbf{0} \\ \mathbf{0} & \sigma_a \end{pmatrix}, \qquad \eta_a = \frac{\mathrm{i}}{2}\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \sigma_a & \mathbf{0} \end{pmatrix} \qquad \text{for} \quad D_2\left(\frac{1}{2}\right). \tag{9}$$

Here $\sigma_a$ are the Pauli matrices and $\mathbf{0}$ is a $2 \times 2$ zero matrix.

Realization (8) with conditions (7) yields equation (1) trivial, i.e., with zero $\beta$-matrices.

The elements of the carrier space of representation (9) will be called the Galilean bi-spinors. It can be found in [5] how the Galilean bi-spinors transform w.r.t. the finite transformations from the Galilei group.

Solutions of relations (7) with $S_a, \eta_a$ given by formulae (9) can be written as

$$\beta_0 = \begin{pmatrix} \mathtt{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \qquad \beta_a = \begin{pmatrix} \mathbf{0} & \sigma_a \\ \sigma_a & \mathbf{0} \end{pmatrix}, \qquad \beta_4 = \begin{pmatrix} \kappa\mathtt{I} & -\mathrm{i}\omega\mathtt{I} \\ \mathrm{i}\omega\mathtt{I} & 2\mathtt{I} \end{pmatrix}, \qquad a = 1, 2, 3, \tag{10}$$

where $\mathtt{I}$ and $\mathbf{0}$ are the $2 \times 2$ unit and zero matrices respectively, and $\omega$ and $\kappa$ are constant multipliers.

Note that parameter $\kappa$ can be chosen zero since the transformation $\Psi \to e^{i\kappa mt}\Psi$ leaves equation (1) invariant. Parameter $\omega$ is inessential too since it can be annulled by the transformation $\beta_\mathtt{m} \to U^\dagger \beta_\mathtt{m} U$, where $\mathtt{m} = 0, 1, 2, 3, 4$ and

$$U = \begin{pmatrix} \mathtt{I} & -i\omega\mathtt{I} \\ \mathbf{0} & \mathtt{I} \end{pmatrix}.$$

Let us note that if we consider a more general case in which matrices $S_a$ and $\eta_a$ are represented by a direct sum of an arbitrary finite number of matrices (9) and solve the related equations (7), then we obtain matrices $\beta_\mathtt{m}$ which can be reduced to direct sums of matrices (10) and zero matrices. In other words, equation (1) with matrices (10) is the only non-decoupled system of the first-order equations for spin 1/2 field invariant under the Galilei group.

Equation (1) with matrices (10) and $\omega = \kappa = 0$ coincides with the Lévy-Leblond equation in [3].

Let us remark that matrices $\hat{\gamma}_\mathtt{n} = \eta\beta_\mathtt{n}|_{\kappa=\omega=0}$, $\mathtt{n} = 0, 1, 2, 3, 4$ with

$$\eta = \begin{pmatrix} \mathbf{0} & \mathtt{I} \\ \mathtt{I} & \mathbf{0} \end{pmatrix} \tag{11}$$

satisfy the following relations:

$$\hat{\gamma}_\mathtt{n}\hat{\gamma}_\mathtt{m} + \hat{\gamma}_\mathtt{m}\hat{\gamma}_\mathtt{n} = 2\hat{g}_\mathtt{nm}, \tag{12}$$

where $\hat{g}_\mathtt{nm}$ is a symmetric tensor whose nonzero components are

$$\hat{g}_{04} = \hat{g}_{40} = -\hat{g}_{11} = -\hat{g}_{22} = -\hat{g}_{33} = 1. \tag{13}$$

In the Galilei-invariant approach tensor (13) plays the same role as the metric tensor (50) for the Minkovski space in relativistic theory.

The matrices $\hat{\gamma}_\mathtt{m}$ will be used many times later on. Therefore, for convenience, we present them explicitly, namely

$$\hat{\gamma}_0 = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathtt{I} & \mathbf{0} \end{pmatrix}, \qquad \hat{\gamma}_a = \begin{pmatrix} \mathbf{0} & -\sigma_a \\ \sigma_a & \mathbf{0} \end{pmatrix}, \qquad \hat{\gamma}_4 = \begin{pmatrix} \mathbf{0} & 2\mathtt{I} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \qquad a = 1, 2, 3. \tag{14}$$

### *2.3. Scalar and vector fields and the corresponding wave equations*

*2.3.1. The indecomposable representations for scalar and vector fields.* A complete description of indecomposable representations of the algebra $hg(1, 3)$ in the spaces of vector and scalar fields is given in [5]. The corresponding matrices $S_a$ and $\eta_a$ have the following forms:

$$S_a = \begin{pmatrix} \mathtt{I}_{n\times n} \otimes s_a & \cdot \\ \cdot & \mathbf{0}_{\mathbf{m}\times\mathbf{m}} \end{pmatrix}, \qquad \eta_a = \begin{pmatrix} A_{n\times n} \otimes s_a & B_{n\times m} \otimes k_a^\dagger \\ C_{m\times n} \otimes k_a & \mathbf{0}_{\mathbf{m}\times\mathbf{m}} \end{pmatrix}, \tag{15}$$

where $\mathtt{I}_{n\times n}$ and $\mathbf{0}_{\mathbf{m}\times\mathbf{m}}$ are unit and zero matrices of dimension $n \times n$ and $m \times m$, respectively, $A_{n\times n}$, $B_{n\times m}$ and $C_{m\times n}$ are matrices of indicated dimensions whose forms will be specified later on, $s_a$ are matrices of spin one with elements $(s_a)_{bc} = i\varepsilon_{abc}$ and $k_a$ are $1 \times 3$ matrices of the form

$$k_1 = (i, 0, 0), \qquad k_2 = (0, i, 0), \qquad k_3 = (0, 0, i). \tag{16}$$

Matrices (15) fulfil relations (6), iff matrices $A_{n\times n}$, $B_{n\times m}$ and $C_{m\times n}$ satisfy the following relations (we have omitted the related subindices):

$$AB = 0, \qquad CA = 0, \qquad A^2 + BC = 0. \tag{17}$$

This system of matrix equations appears to be completely solvable, i.e. it is possible to find all non-equivalent indecomposable matrices $A$, $B$ and $C$ which satisfy relations (17). Any set

**Table 1.** Solutions of equations (17).

| No | $(n, k, \lambda)$ | Matrices $A, B, C$ |
|---|---|---|
| 1 | (0, 1, 0) | $A$, $B$ and $C$ do not exist since $n = 0$ |
| 2 | (1, 0, 0) | $A = 0$, $B$ and $C$ do not exist since $k = 0$ |
| 3 | (1, 1, 0) | $A = 0, B = 0, C = 1$ |
| 4 | (1, 1, 1) | $A = 0, B = 1, C = 0$ |
| 5 | (1, 2, 1) | $A = 0, B = (10), C = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ |
| 6 | (2, 0, 0) | $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $B$ and $C$ do not exist since $k = 0$ |
| 7 | (2, 1, 0) | $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, C = (10)$ |
| 8 | (2, 1, 1) | $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, C = (00)$ |
| 9 | (2, 2, 1) | $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ |
| 10 | (3, 1, 1) | $A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, C = (1\ \ 0\ \ 0)$ |

of such matrices generates a representation of the algebra $hg(1, 3)$ whose basis elements are of the forms specified in (15).

According to [5] indecomposable representations $D(n, m, \lambda)$ of $hg(1, 3)$ for scalar and vector fields are labelled by integers $n, k$ and $\lambda$. They specify dimensions of submatrices in (15) and the rank of matrix $B$, respectively. As shown in [5], there exist ten non-equivalent indecomposable representations $D(n, k, \lambda)$ of $hg(1, 3)$ which correspond to matrices $A_{n\times n}$, $B_{n\times k}$ and $C_{k\times n}$ given in table 1.

In addition to the scalar representation whose generators are written in (15) and in item 1 of table 1, there exist nine vector representations corresponding to matrices enumerated in table 1, items 2–10. The corresponding basis elements are matrices of dimension $(3n + k) \times (3n + k)$ whose explicit forms are given in (15) and in table 1.

The finite Galilei transformations of vector fields (which can be obtained by integrating the Lie equations for generators (15)) and examples of such fields can be found in paper [5].

*2.3.2. General wave equations for vector and scalar fields.* Let us consider equation (1) and describe all admissible matrices $\beta_4$ compatible with the invariance conditions (7). We shall restrict ourselves to matrices $\eta_a, S_a$ belonging to the representations described in subsection 2.3.1 (see table 1) or to direct sums of these representations. Then the general form of matrices $S_a$ and $\eta_a$ is again given by equations (15) where, however, matrices $A, B$ and $C$ can be reducible

$$A = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \cdot & \\ & & & \cdot \\ & & & & \cdot \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & & & \\ & B_2 & & \\ & & \cdot & \\ & & & \cdot \\ & & & & \cdot \end{pmatrix}, \quad C = \begin{pmatrix} C_1 & & & \\ & C_2 & & \\ & & \cdot & \\ & & & \cdot \\ & & & & \cdot \end{pmatrix} \tag{18}$$

and are of dimensions $N \times N$, $M \times N$ and $N \times M$, respectively, with $N$ and $M$ being arbitrary integers. The unit and zero matrices in the associated spin operator $\mathbf{S}$ defined by equation (15) are $(N \times N)$- and $(M \times M)$-dimensional, respectively.

The sets of matrices $(A_1, B_1, C_1), (A_2, B_2, C_2), \ldots$ are supposed to be indecomposable sets presented in table 1. Any of them is labelled by a multi-index $q_i = (n_i, k_i, \lambda_i), i = 1, 2, \ldots$.

Matrices $\beta_4$ and $\beta_0$ must commute with $\mathbf{S}$ and therefore have the following block diagonal form:

$$\beta_4 = \begin{pmatrix} R_{N\times N} & \mathbf{0_{N\times M}} \\ \mathbf{0_{M\times N}} & E_{M\times M} \end{pmatrix}, \qquad \beta_0 = \begin{pmatrix} F_{N\times N} & \mathbf{0_{N\times M}} \\ \mathbf{0_{M\times N}} & G_{M\times M} \end{pmatrix}. \tag{19}$$

Let us denote by $|q, s, \nu\rangle$ a vector belonging to a carrier space of the representation $D_q$ of the algebra $hg(1, 3)$, where $q = (n, k, \lambda)$ is a multi-index which labels a particular indecomposable representation as indicated in table 1, $s$ is a spin quantum number which is equal to 0, 1 and index $\nu$ specifies degenerate subspaces with the same fixed $s$. Then taking into account that matrix $\beta_4$ commutes with $S_a$ its elements can be written as

$$\langle q, s, \nu|\beta_4|q', s', \nu'\rangle = \delta_{s1}\delta_{s'1}R_{\nu\nu'}(q, q') + \delta_{s0}\delta_{s'0}E_{\nu\nu'}(q, q'). \tag{20}$$

In order to find matrices $R(q, q')$ and $E(q, q')$ (whose elements are denoted by $R_{\nu\nu'}$ and $E_{\nu\nu'}$, respectively) expression (20) has to be substituted into (7) and matrices $\eta$ in form (15) together with relations (17) used. As a result we obtain the following condition:

$$(A^\dagger)^2R + R(A')^2 = A^\dagger RA' - C^\dagger EC', \tag{21}$$

where $A, C$ (and $A', C'$) are submatrices used in (17), which correspond to representation $D_q$ (and $D_{q'}$).

Formulae (20) and (21) express all necessary and sufficient conditions for matrix $\beta_4$ imposed by the Galilei invariance conditions (7). Suppose matrix $\beta_4$ in (20) which satisfies (21) be known, then the remaining matrices $\beta_a$ $(a = 1, 2, 3)$ and $\beta_0$ can be found by direct use of the first and second relations in (7). In this way we obtain

$$\begin{aligned}
&\langle q, s, \lambda|\beta_0|q', s', \lambda'\rangle = \delta_{s1}\delta_{s'1}F_{\lambda\lambda'}(q, q') + \delta_{s0}\delta_{s'0}G_{\lambda\lambda'}(q, q'),\\
&\langle q, s, \lambda|\beta_a|q', s', \lambda'\rangle = \mathrm{i}\big(\delta_{s1}\delta_{s'1}H_{\lambda\lambda'}(q, q')s_a + \delta_{s1}\delta_{s'0}M_{\lambda\lambda'}(q, q')k_a^\dagger - \delta_{s0}\delta_{s'1}M^\dagger_{\lambda\lambda'}(q, q')k_a\big),
\end{aligned} \tag{22}$$

where $s_a$ are matrices of spin one, $k_a$ are matrices (16) and $F(q, q')$, $G(q, q')$, $H(q, q')$ and $M(q, q')$ are matrices defined by the following relations:

$$\begin{aligned}
&H = A^\dagger R - RA', \qquad M = C^\dagger E - RB', \qquad F = C^\dagger EC' + A^\dagger RA',\\
&G = 2B^\dagger RB' - B^\dagger C^\dagger E - EC'B'.
\end{aligned} \tag{23}$$

Thus, in order to derive a Galilei-invariant equation (1) for vector fields, it is sufficient to choose a realization of the algebra $hg(1, 3)$ from table 1 or a direct sum of such realizations and find the associated matrix $\beta_4$ (20) whose block matrices $R$ and $E$ satisfy relations (21). Then the corresponding matrices $\beta_0$ and $\beta_a$ are determined via relations (22) and (23).

All non-trivial solutions of matrices $R$ and $E$ are specified in the appendix. Thus formulae (20)–(23) and tables A1–A3 in appendix A determined all possible matrices $\beta_4$ and $\beta_\mu$ which define the Galilei-invariant equations (1) for fields of spin 1.

Note that all these equations admit a Lagrangian formulation with Lagrangians in the following standard form

$$L = \tfrac{1}{2}\Psi^\dagger(\beta_\mu p^\mu + \beta_4 m)\Psi + \text{h.c.} \tag{24}$$

*2.3.3. Consistency conditions.* In the previous section we have found all matrices $\beta_{\mathtt{m}}$ for which equation (1) is invariant with respect to vector and scalar representations of the homogeneous Galilei group. However, the Galilei invariance itself guarantees neither the consistency nor the right number of independent components. Moreover, the spin content of each of the obtained equations as well as their possibilities of describing fundamental quantum-mechanical systems have not yet been discussed.

In this subsection we present further constraints on matrices $\beta_{\mathtt{m}}$ which should be imposed in order to obtain a consistent equation for a system with a fixed spin.

In this context, a non-relativistic quantum system is said to be *fundamental* if the space of its states forms a carrier space of an irreducible representation of the Galilei group $G(1,3)$. We shall call such systems 'non-relativistic particles' or simply 'particles'.

A non-relativistic quantum system is said to be *composed* provided the space of its states forms a carrier space of some reducible representation of $G(1,3)$.

First we consider equations (5) for fundamental systems. Note that for the group $G(1,3)$ there exist the following three invariant operators:

$$C_1 = M, \qquad C_2 = 2MP_0 - \mathbf{P}^2, \qquad \text{and} \qquad C_3 = (M\mathbf{J} - \mathbf{P}\times\mathbf{G})^2. \tag{25}$$

Here, $M$ and $P_0$ are scalars and $\mathbf{P}$, $\mathbf{J}$ and $\mathbf{G}$ are three vectors whose components are specified by equations (5). Eigenvalues of the operators $C_1$, $C_2$ and $C_3$ are associated with mass, internal energy and with square of mass operator multiplied by eigenvalues of total spin, respectively.

Galilei-invariant equation (1) is said to be *consistent* and describes a particle with mass $m$, internal energy $\varepsilon$ and spin $s$ if it has non-trivial solutions $\Psi$ which form a (non-degenerate) carrier space of some representation of the Galilei group on which the following conditions are true:

$$C_1\Psi = m\Psi, \qquad C_2\Psi = \varepsilon\Psi \qquad \text{and} \qquad C_3\Psi = m^2 s(s+1)\Psi. \tag{26}$$

Moreover, for a fundamental particle the spin value $s$ is fixed which corresponds to an irreducible representation, while for a composed system the corresponding spin operator has two values: $s = 0$ and $s = 1$ since the associated representation is reducible.

In accordance with the above definition of consistency, the number of independent components of function $\Psi$, satisfying a consistent equation (1), must be equal to the number of spin degrees of freedom, namely to $2s + 1$ for a fundamental particle and to four for a composed one.

Now we shall show that relations (26) generate extra conditions for $\beta$-matrices so that equation (1) with such $\beta_{\mathtt{m}}$ guarantees the validity of equations (26).

Function $\Psi$ satisfying equation (1) cotransforms according to a particular representation of the Galilei group whose infinitesimal generators are specified in (5). Using (5) we find the following forms of invariant operators (26):

$$\begin{aligned} &C_1 = Im, \qquad C_2 = 2mp_0 - \mathbf{p}^2,\\ &C_3 = m^2\mathbf{S}^2 + m(\mathbf{S}\times\boldsymbol{\eta})\cdot\mathbf{p} - m(\boldsymbol{\eta}\times\mathbf{S})\cdot\mathbf{p} + \mathbf{p}^2\boldsymbol{\eta}^2 - (\mathbf{p}\cdot\boldsymbol{\eta})^2. \end{aligned} \tag{27}$$

We see that $C_3$ is a rather complicated second-order differential operator with matrix coefficients. In order to diagonalize this operator, we apply the similarity transformation

$$\Psi \to \Psi' = W\Psi, \qquad C_a \to C'_a = WC_aW^{-1}, \qquad a = 1, 2, 3, \tag{28}$$

with

$$W = \exp\left(\frac{\mathrm{i}}{m}\boldsymbol{\eta}\cdot\mathbf{p}\right). \tag{29}$$

Let us remark that since $(\boldsymbol{\eta}\cdot\mathbf{p})^3 = 0$ for representations $D(3,1,1)$ and $D(1,2,1)$ and $(\boldsymbol{\eta}\cdot\mathbf{p})^2 = 0$ for the remaining representations described in subsection 2.3.1, $W$ is the second- or the first-order differential operator in $\mathbf{x}$.

Using conditions (6) we find that

$$C'_1 = C_1, \qquad C'_2 = C_2 \qquad \text{and} \qquad C'^2_3 = \mathbf{S}^2. \tag{30}$$

Function $\Psi'$ has to satisfy conditions (26) with the transformed invariant operators (30), from which follow that

$$(2mp_0 - \mathbf{p}^2)\Psi' = \varepsilon\Psi' \tag{31}$$

and

$$\mathbf{S}^2\Psi' = s(s+1)\Psi'. \tag{32}$$

In order to see when conditions (31) and (32) are true we transform equation (1) by means of $W$. We obtain

$$(\beta_0 C_2 + \beta_4 2m^2)\Psi' = 0 \tag{33}$$

since, in accordance with (7), we have

$$2m(W^{-1})^{\dagger}(\beta^{\mu} p_{\mu} + \beta_4 m)W^{-1} = \beta_0(2mp_0 - \mathbf{p}^2) + \beta_4 2m^2.$$

Equation (33) in order to be compatible with (31) implies that the matrix $\beta_0\varepsilon + \beta_4 2m^2$ (where $\varepsilon$ is an eigenvalue of the Casimir operator $C_2$) should be non-regular for some particular value of $\varepsilon$. Moreover, solutions of equation (33) must also satisfy condition (32) and form a carrier space of irreducible representation $D(s)$ of the rotation group; consequently, equation (33) must have $(2s + 1)$ independent solutions. Let us find now restrictions on matrices $\beta_0$ and $\beta_4$ generated by these solutions.

As shown in section 2.3.1, both matrices $\beta_0$ and $\beta_4$ have the block diagonal form given by equation (19). Thus equation (33) is decoupled to two subsystems

$$(RC_2 + 2m^2F)\varphi_1 = 0, \tag{34}$$

and

$$(EC_2 + 2m^2G)\varphi_2 = 0, \tag{35}$$

where the functions $\varphi_1$ and $\varphi_2$ denote eigenfunction columns with $3n$ and $k$ components respectively so that $\Psi' = \mathtt{column}\,(\varphi_1, \varphi_2)$.

Equations (31) and (32) are consequences of (34) and (35) provided determinants of matrices $R$, $F$, $E$ and $G$ satisfy the following conditions:

$$\mathtt{det}(RC_2 + 2m^2F) = \nu C_2 + \mu, \qquad \mathtt{det}(EC_2 + 2m^2G) = \nu' C_2 + \mu', \tag{36}$$

where $\mu$, $\nu$ and $\mu'$, $\nu'$ are numbers satisfying one of the following relations:

$$\nu \neq 0, \quad \mu = -\varepsilon\nu, \quad \nu' = 0, \quad \mu' \neq 0 \quad \text{for} \quad s = 1, \tag{37}$$

$$\nu = 0, \quad \mu \neq 0, \quad \nu' \neq 0, \quad \mu' = -\varepsilon\nu' \quad \text{for} \quad s = 0. \tag{38}$$

Thus to find Galilei-invariant equations (1) for a particle with a fixed spin we can start with equations described in the previous section and impose conditions (36) on block components of matrices $\beta_0$ and $\beta_4$.

In an analogous way we find the consistency conditions for equations (1) describing composite systems. In this case relations (31) and (32) should be valid too. However, $s$ can take two values, $s = 1$ and $s = 0$. We suppose that these spin states are non-degenerate, so that function $\Psi'$ should have four independent components—three of them corresponding to spin one and one to spin zero state. Then conditions for determinants of matrices $FC_2 + 2m^2R$ and $GC_2 + 2m^2E$ again have form (36) where, however,

$$\nu\nu' \neq 0, \qquad \mu = -\varepsilon\nu, \qquad \mu' = -\varepsilon\nu'. \tag{39}$$

In the following sections we consider examples of various Galilean equations (1) in more detail.

## 3. Special classes of the Galilean wave equations for free fields

### 3.1. Equations invariant with respect to the indecomposable representations

In sections 2.3.2 and 2.3.3 we have given a general description of all possible Galilei wave equations (1) for vector and scalar fields. Now we shall present an analysis of these equations in detail. In this and following sections we shall consider equations for systems with both one fixed spin state and two spin values states.

First we restrict ourselves to the indecomposable representations of the algebra $hg(1,3)$ specified in equation (15) and table 1, and find the associated matrices $\beta_4$, $\beta_0$ and $\beta_a$ which appear in the Galilei-invariant equations (1). Taking into account that $A' = A$, $B' = B$ and $C' = C$ in (21), where $A$, $B$ and $C$ are matrices given in table 1, and using the results present in tables A1–A3 in appendix A, we easily find the associated block matrices $R$, $E$ and consequently all block matrices (23) and matrices (20) and (22). In order to simplify matrices $\beta_{\mathtt{m}}$ $(\mathtt{m} = 0, 1, \ldots, 4)$ we use the transformations

$$\beta_{\mathtt{m}} \to V^{\dagger}\beta_{\mathtt{m}}V, \tag{40}$$

where $V$ are invertible matrices commuting with the Galilei boost generators $\eta_a$.

It turns out that non-trivial solutions for $\beta_4$ (and consequently also for $\beta_0$ and $\beta_a$) exist only for representations $D(1,1,0)$, $D(2,1,0)$, $D(2,2,1)$ and $D(3,1,1)$. They have the form:

for representation $D(1,1,0)$

$$\beta_4 = \begin{pmatrix} \mathtt{I}_{3\times3} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & 0 \end{pmatrix}, \qquad \beta_0 = \begin{pmatrix} \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & 2 \end{pmatrix}, \qquad \beta_a = \mathrm{i}\begin{pmatrix} \mathbf{0}_{3\times3} & -k_a^{\dagger} \\ k_a & 0 \end{pmatrix}; \tag{41}$$

for representation $D(2,1,0)$

$$\beta_4 = \begin{pmatrix} \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{3\times3} & \mathtt{I}_{3\times3} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & 1 \end{pmatrix}, \qquad \beta_0 = \begin{pmatrix} 2\mathtt{I}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & 0 \end{pmatrix},$$

$$\beta_a = \mathrm{i}\begin{pmatrix} \mathbf{0}_{3\times3} & s_a & k_a^{\dagger} \\ -s_a & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1} \\ -k_a & \mathbf{0}_{1\times3} & 0 \end{pmatrix}; \tag{42}$$

for representation $D(2,2,1)$ :

$$\beta_4 = \begin{pmatrix} \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{3\times3} & \mathtt{I}_{3\times3} & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & 0 & 0 \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & 0 & 1 \end{pmatrix}, \qquad \beta_0 = \begin{pmatrix} 2\mathtt{I}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & 2 & 0 \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & 0 & 0 \end{pmatrix},$$

$$\beta_a = \mathrm{i}\begin{pmatrix} \mathbf{0}_{3\times3} & s_a & \mathbf{0}_{3\times1} & k_a^{\dagger} \\ -s_a & \mathbf{0}_{3\times3} & -k_a^{\dagger} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & k_a & 0 & 0 \\ -k_a & \mathbf{0}_{1\times3} & 0 & 0 \end{pmatrix}; \tag{43}$$

and finally for representation $D(3,1,1)$ :

$$\beta_4 = \begin{pmatrix} \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \nu\mathtt{I}_{3\times3} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{3\times3} & \nu\mathtt{I}_{3\times3} & \mathtt{I}_{3\times3} & \mathbf{0}_{3\times1} \\ \nu\mathtt{I}_{3\times3} & \mathtt{I}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & -\nu \end{pmatrix}, \qquad \beta_0 = \begin{pmatrix} \mathbf{0}_{3\times3} & \mathtt{I}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1} \\ \mathtt{I}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & 0 \end{pmatrix},$$

$$\beta_a = \mathrm{i}\begin{pmatrix} \mathbf{0_{3\times3}} & \mathbf{0_{3\times3}} & s_a & \mathbf{0_{3\times1}} \\ \mathbf{0_{3\times3}} & \mathbf{0_{3\times3}} & \mathbf{0_{3\times3}} & -k_a^\dagger \\ -s_a & \mathbf{0_{3\times3}} & \mathbf{0_{3\times3}} & \mathbf{0_{3\times1}} \\ \mathbf{0_{1\times3}} & k_a & \mathbf{0_{1\times3}} & 0 \end{pmatrix}. \tag{44}$$

Here, $\mathtt{I}_{k\times k}$ and $\mathbf{0_{k\times r}}$ are unit and zero matrices of dimensions $k \times k$ and $k \times r$, respectively and $\nu$ is an arbitrary non-vanishing parameter.

Thus there are four equations (1) for spinor and vector fields which are invariant with respect to the above-mentioned indecomposable representations of the homogeneous Galilei group. Their associated matrices $\beta_\mu$ and $\beta_4$ are given by formulae (41)–(44).

Equations (1) with $\beta$-matrices specified in (41), (42) and (44) are equivalent to those discussed in papers [20–22], respectively. However, equation (1) with $\beta$-matrices of the form (43) is to the best of our knowledge new. Note that the related submatrices $F$, $R$ and $G$, $E$ satisfy relations (36) and (39), so the associate equation describes a Galilean quantum-mechanical system whose spin can take two values: $s = 1$ and $s = 0$.

Considering matrices (44) we conclude that the corresponding matrices

$$\tilde\beta_\mu = \eta\beta_\mu, \qquad \mu = 0, 1, 2, 3, \qquad \text{and} \qquad \tilde\beta_4 = \eta\beta_4 - \nu\mathtt{I}_{10\times10}, \tag{45}$$

where $\eta$ is an invertible matrix

$$\eta = \begin{pmatrix} \mathbf{0_{3\times3}} & \mathbf{0_{3\times3}} & \mathtt{I}_{3\times3} & \mathbf{0_{3\times1}} \\ \mathbf{0_{3\times3}} & \mathtt{I}_{3\times3} & \mathbf{0_{3\times3}} & \mathbf{0_{3\times1}} \\ \mathtt{I}_{3\times3} & \mathbf{0_{3\times3}} & \mathbf{0_{3\times3}} & \mathbf{0_{3\times1}} \\ \mathbf{0_{1\times3}} & \mathbf{0_{1\times3}} & \mathbf{0_{1\times3}} & -1 \end{pmatrix} \tag{46}$$

satisfying the following relations:

$$\beta_\mu\beta_\nu\beta_\sigma + \beta_\sigma\beta_\nu\beta_\mu = 2(g_{\mu\nu}\beta_\sigma + g_{\sigma\nu}\beta_\mu), \tag{47}$$

where $g_{\mu\nu}$ is a 'Galilean metric tensor' given in (13).

Up to definition of metric tensor $g_{\mu\nu}$ relations (47) coincide with the defining relations for the Duffin–Kemmer–Petiau (DKP) algebra [25]. Following [26] we say that such relations define a Galilean DKP algebra.

Equation (1) with matrices $\tilde\beta_\mu$, $\tilde\beta_4$ which satisfy the Galilean DKP algebra is called the *Galilean Duffin–Kemmer equation*. This equation was considered for the first time apparently in [21].

There exist also a number of wave equations (1) invariant with respect to *decomposable* representations of $hg(1, 3)$. We shall discuss some of them in the subsections which follow.

### *3.2. The Galilean Proca equations*

In addition to equations discussed in the previous section (whose solutions belong to carrier spaces of indecomposable representations of the homogeneous Galilei algebra), there exist also a number of wave equations (1) invariant with respect to decomposable representations of $hg(1, 3)$. Complete description of such equations is given in section 2.3.2 and appendix A. In this section we shall discuss some of them as examples.

Let us consider first wave equations whose solutions are vectors from a carrier space of the direct sum of representations $D(3, 1, 1) \oplus D(2, 1, 1)$ of the algebra $hg(1, 3)$. The associated matrices $\beta_4$, $\beta_0$ and $\beta_a$ can be found using relations (20), (22), (23) and tables A1 and A2 where submatrices $R(q, q)$, $R(q, q')$, $R(q', q')$ and $E(q, q)$, $E(q, q')$, $E(q', q')$ are specified for $q = (3, 1, 1)$ and $q' = (1, 2, 1)$. We can simplify these matrices by using the equivalence transformations (40).

In this way we realize that there are two consistent equations for the representation $D(3,1,1) \oplus D(2,1,1)$: one describing a particle with spin $s = 1$ and the other describing a composed system with spins $s = 1$ and $s = 0$. Here we present a covariant formulation of these equations.

The equation for a particle with spin $s = 1$ can be written in the following form:

$$p^{\mathtt{k}}\Psi^{\mathtt{n}} - p^{\mathtt{n}}\Psi^{\mathtt{k}} = m\Psi^{\mathtt{kn}}, \qquad p_{\mathtt{k}}\Psi^{\mathtt{nk}} = \lambda\delta^{\mathtt{n}4}m\Psi^4. \tag{48}$$

Here, $\delta^{\mathtt{nk}}$ is the Kronecker symbol, $\Psi^{\mathtt{nk}}$ $(\mathtt{n}, \mathtt{k} = 0, 1, 2, 3, 4)$ is an antisymmetrical tensor and $\Psi^{\mathtt{n}}$ is a 5-vector which transform in accordance with representations $D(3,1,1)$ and $D(1,2,1)$, respectively.

Equation (48) includes an arbitrary real parameter $\lambda \neq 0$ whose value cannot be fixed using only conditions of the Galilei invariance.

The equation for a system with two spin states involves again tensor and 5-vector variables which satisfy the following system:

$$p^{\mathtt{k}}\Psi^{\mathtt{n}} - p^{\mathtt{n}}\Psi^{\mathtt{k}} = m\Psi^{\mathtt{kn}}, \qquad p_{\mathtt{k}}\Psi^{\mathtt{nk}} = \nu m\Psi^{\mathtt{n}}. \tag{49}$$

The Galilei-invariant equations (48), (49) are quite similar to the relativistic Proca equations (2), but with the following differences:

- all indices $\mathtt{m}, \mathtt{n}$ in (48) take values $0, \ldots, 4$ while in (2) we have $\mu, \nu = 0, 1, 2, 3$;
- the relativistic 4-gradient $p_\nu$ is replaced by the Galilean 5-vector $p = (p^0, p^1, p^2, p^3, p^4)$, where $p^0 = \mathrm{i}\frac{\partial}{\partial t}$, $p^a = \mathrm{i}\frac{\partial}{\partial x_a}$, $a = 1, 2, 3$, and $p^4 = m$;
- covariant indices $\mathtt{m}, \mathtt{n}$ are raised and lowered by using the Galilean metric tensor (13) instead of the relativistic metric tensor
$$g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1); \tag{50}$$
- finally, equation (49) describes a system with two spin states, while both the relativistic Proca equation and the Galilei-invariant equation (48) describe a particle with fixed spin $s = 1$. We shall call (48) *the Galilean first-order Proca equation*.

Let us stress here that the considered Galilean Proca equations are not non-relativistic approximations of the relativistic Proca equation (2 ) (see appendix C where a contraction of (2) to its non-relativistic approximation is shown).

Substituting the expression for $\Psi^{\mathtt{nk}}$ from the first into the second equation in (48), we obtain the *Galilean second-order Proca equation*

$$p_{\mathtt{n}}p^{\mathtt{n}}\Psi^{\mathtt{m}} - p^{\mathtt{m}}p_{\mathtt{n}}\Psi^{\mathtt{n}} + \lambda\delta^{\mathtt{m}0}m^2\Psi^4 = 0 \tag{51}$$

where $\mathtt{m}, \mathtt{n} = 0, 1, 2, 3, 4$. Equation (51) admits, like (48), a Lagrangian formulation and describes a particle with spin 1. The corresponding Lagrangian has the form

$$L = (p_{\mathtt{m}}\Psi_{\mathtt{n}} - p_{\mathtt{n}}\Psi_{\mathtt{m}})^*(p^{\mathtt{m}}\Psi^{\mathtt{n}} - p^{\mathtt{n}}\Psi^{\mathtt{m}}) + (p^{\mathtt{m}}\Psi_{\mathtt{m}})^* p_{\mathtt{n}}\Psi^{\mathtt{n}} - (p_{\mathtt{m}}\Psi_{\mathtt{n}})^* p^{\mathtt{m}}\Psi^{\mathtt{n}} - \lambda m^2\Psi_0^*\Psi^4, \tag{52}$$

where the asterisk $*$ denotes complex conjugation.

### 3.3. The Galilean Rarita–Schwinger equation

Till now we have used our knowledge of the indecomposable representations of the algebra $hg(1,3)$ for spinor, scalar and vector fields to construct wave equations for fields of spin $\tilde{s} \leqslant 1$. In this section we derive Galilean invariant equations for the field transforming as a direct product of spin 1/2 and spin 1 fields. The relativistic analogue of such a system is the famous Rarita–Schwinger equation.

The relativistic Rarita–Schwinger equation for a particle with spin $s = \frac{3}{2}$ is constructed by using a vector–spinor wavefunction $\Psi^{\mu}_{\alpha}$, where $\mu = 0, 1, 2, 3$ and $\alpha = 1, 2, 3, 4$ are vector and spinor indices, respectively. Moreover, $\Psi^{\mu}_{\alpha}$ is supposed to satisfy the equation

$$(\gamma^{\nu}p_{\nu} - m)\Psi^{\mu} - \gamma^{\mu}p_{\nu}\Psi^{\nu} - p^{\mu}\gamma_{\nu}\Psi^{\nu} + \gamma^{\mu}(\gamma_{\nu}p^{\nu} + m)\gamma_{\sigma}\Psi^{\sigma} = 0, \tag{53}$$

where $\gamma^{\mu}$ are the Dirac matrices acting on the spinor index $\alpha$ of $\Psi^{\mu}_{\alpha}$ which we have omitted.

Reducing the left-hand side of equation (53) by $p_{\mu}$ and $\gamma_{\mu}$ we obtain the following expressions:

$$\gamma_{\mu}\Psi^{\mu} = 0, \qquad \texttt{and} \qquad p_{\mu}\Psi^{\mu} = 0, \tag{54}$$

which reduce the number of independent components of $\Psi^{\mu}_{\alpha}$ to 8 as required for a wavefunction of a relativistic particle with spin $3/2$.

Using our knowledge of invariants for the Galilean vector fields from [5] and [6] we can easily find a Galilean analogue of equation (53). Like in the case of the Galilean Proca equation we begin with a 5-vector $\Psi^{\mathtt{m}}$, $\mathtt{m} = 0, 1, 2, 3, 4$ which has, in addition, a bi-spinorial index which we do not write explicitly. Thus our Galilei-invariant equation can be written in the following form:

$$\hat{\gamma}_{\mathtt{n}}p^{\mathtt{n}}\hat{\Psi}^{\mathtt{m}} - \hat{\gamma}^{\mathtt{m}}p_{\mathtt{n}}\Psi^{\mathtt{n}} - p^{\mathtt{m}}\hat{\gamma}_{\mathtt{n}}\Psi^{\mathtt{n}} + \hat{\gamma}^{\mathtt{m}}\hat{\gamma}_{\mathtt{n}}p^{\mathtt{n}}\hat{\gamma}_{\mathtt{r}}\hat{\Psi}^{\mathtt{r}} + \lambda\delta^{\mathtt{m}0}m\hat{\Psi}^{4} = 0. \tag{55}$$

Here, $\hat{\gamma}_{\mathtt{n}}$ are the Galilean $\gamma$-matrices (14), $p^{\mathtt{m}}$ is a Galilean '5-momentum' defined in the previous subsection (see the second item there), $\lambda$ is an arbitrary non-vanishing parameter and raising and lowering of indices $\mathtt{m}$ and $\mathtt{n}$ is done by using the Galilean metric tensor (13).

Like equation (51) equation (55) admits a Lagrangian formulation. The corresponding Lagrangian can be written as

$$L = \tfrac{1}{2}\big(\bar{\hat{\Psi}}_{\mathtt{m}}\hat{\gamma}_{\mathtt{n}}p^{\mathtt{n}}\hat{\Psi}^{\mathtt{m}} - \bar{\hat{\Psi}}_{\mathtt{m}}\hat{\gamma}^{\mathtt{m}}p_{\mathtt{n}}\hat{\Psi}^{\mathtt{n}} - \bar{\hat{\Psi}}_{\mathtt{m}}p^{\mathtt{m}}\hat{\gamma}_{\mathtt{n}}\hat{\Psi}^{\mathtt{n}} + \bar{\hat{\Psi}}_{\mathtt{m}}\hat{\gamma}^{\mathtt{m}}\hat{\gamma}_{\mathtt{n}}p^{\mathtt{n}}\hat{\gamma}_{\alpha}\hat{\Psi}^{\alpha} + \lambda m\bar{\hat{\Psi}}_{0}\hat{\Psi}^{4}\big) + \text{h.c.}, \tag{56}$$

where $\bar{\hat{\Psi}}_{\mathtt{m}} = \hat{\Psi}^{\dagger}_{\mathtt{m}}\eta$ and $\eta$ is the hermitizing matrix (11).

We prove in appendix B that equation (55) indeed describes a particle with spin $3/2$.

## 4. Equations for charged particles interacting with an external gauge field

### *4.1. Minimal interaction with an external field*

We have described Galilei-invariant equations (1) for free particles with spins 0, $1/2$, 1 and $3/2$. These equations have admitted Lagrangian formulation (24), so that to generalize them to the case of particles interacting with an external field means, as usually, to apply the minimal interaction principle, i.e., to make the following change in the Lagrangian:

$$p^{\mu} \to \pi^{\mu} = p^{\mu} - eA^{\mu}, \tag{57}$$

where $A^{\mu}$ are components of a vector potential of the external field, and $e$ is a particle charge.

Thus we obtain the Lagrangian

$$L = \tfrac{1}{2}\Psi^{\dagger}(\mathbf{x}, t)\big(\beta_{\mu}\pi^{\mu} + \beta_{4}m\big)\Psi(\mathbf{x}, t) + h.c. \tag{58}$$

It is important to note that change (57) is compatible with the Galilei invariance provided the components $(A^0, A^1, A^2, A^3)$ of the vector potential transform as a Galilean 4-vector, i.e., as

$$A^0 \to A^0 + \mathbf{v}\cdot\mathbf{A}, \qquad \mathbf{A} \to \mathbf{A}. \tag{59}$$

Such potential corresponds to a 'magnetic' limit of the Maxwell equations, see [18], [27]. For other possible Galilei potentials of the electromagnetic field see [7].

The desired equations for a charged particle interacting with an external field are the Euler equations derived from the Lagrangian (58).

However let us note that it is also possible to introduce interaction with an external field via other means, e.g., via an anomalous (Pauli) term.

### *4.2. The Galilean Bhabha equations with minimal and anomalous interactions*

Taking Lagrangian (58) we can derive the following equation for a charged particle interacting with an external field:

$$\left(\beta_\mu\pi^\mu + \beta_4\pi^4\right)\Psi(\mathbf{x}, t) = 0. \tag{60}$$

Let us consider now equation (60) with general matrices $\beta_\mu$ and $\beta_4$. If we restrict ourselves to a vector potential of magnetic type, i.e., to $A = (A^0, \mathbf{A}, 0)$, then

$$\pi^0 = p^0 - eA^0, \qquad \pi^a = p^a - eA^a, \qquad \pi^4 = m. \tag{61}$$

Like free particle equations (1), equation (60) is Galilei-invariant provided matrices $\beta_\mu$, $\beta_4$ satisfy conditions (7). Moreover the vector potential of an external field has to transform according to (59).

Following Pauli [29] we generalize our equation (60) by adding to it an interaction terms linear in an electromagnetic field strength. Then we get the equation

$$\left(\beta_\mu\pi^\mu + \beta_4 m + F\right)\Psi = 0, \tag{62}$$

where

$$F = \frac{e}{m}(\mathbf{A}\cdot\mathbf{H} + \mathbf{G}\cdot\mathbf{E}).$$

Here, $\mathbf{A}$ and $\mathbf{G}$ are matrices determined by requirement of the Galilei invariance, i.e., by demanding that $\mathbf{A}\cdot\mathbf{H}$ and $\mathbf{G}\cdot\mathbf{E}$ have to be Galilean scalars.

In paper [5] we have found the most general form of the Pauli interaction which can be introduced into the Lévy-Leblond equation for a particle of spin 1/2. Finding the general Pauli interaction for other Galilean particles is a special problem for any equation previously considered. Here we restrict ourselves to a systematic analysis of the Pauli terms which is valid for *any* Galilean Bhabha equation.

First we shall prove the following statement.

**Lemma.** *Let $S_a$, $\eta_a$ be matrices which realize a representation of the algebra $hg(1, 3)$, $\Lambda$ be a matrix satisfying the conditions*

$$S_a\Lambda = \Lambda S_a, \qquad \eta_a^\dagger\Lambda = \Lambda\eta_a \tag{63}$$

*and*

$$E_a = \frac{\partial A_0}{\partial x^a} - \frac{\partial A_a}{\partial t}, \qquad H_a = \varepsilon_{abc}\frac{\partial A^b}{\partial x_c}$$

*be vectors of the electric and magnetic field strength, respectively. Then matrices*

$$F_1 = \Lambda(\mathbf{s}\cdot\mathbf{H} - \boldsymbol{\eta}\cdot\mathbf{E}) \qquad \text{and} \qquad F_2 = \Lambda\boldsymbol{\eta}\cdot\mathbf{H} \tag{64}$$

*are invariant with respect to the Galilei transformations provided the vector potential $A$ is transformed in accordance with the Galilean law (59).*

**Proof.** First we note that matrices (64) are scalars with respect to rotations. Then, starting with transformation laws (3) and (59) we easily find that under a Galilei boost the vectors $\mathbf{E}$ and $\mathbf{H}$ co-transform as

$$\mathbf{E} \to \mathbf{E} - \mathbf{v}\times\mathbf{H}, \qquad \mathbf{H} \to \mathbf{H}. \tag{65}$$

On the other hand the transformation laws for matrices $\Lambda\mathbf{S}$ and $\Lambda\boldsymbol{\eta}$ can be found using the exponential mapping of boost generators $\mathbf{G}$ given in equation (5)

$$\begin{aligned}\mathbf{S} &\rightarrow \exp(\mathrm{i}\mathbf{G}^{\dagger}\cdot\mathbf{v})\Lambda\mathbf{S}\exp(-\mathrm{i}\mathbf{G}\cdot\mathbf{v}) = \Lambda\exp(\mathrm{i}\boldsymbol{\eta}\cdot\boldsymbol{v}\ )\mathbf{S}\exp(-\mathrm{i}\boldsymbol{\eta}\cdot\boldsymbol{v}\ ) = \mathbf{s}+\mathbf{v}\times\boldsymbol{\eta},\\ \boldsymbol{\eta} &\rightarrow \exp(\mathrm{i}\mathbf{G}^{\dagger}\cdot\mathbf{v})\Lambda\boldsymbol{\eta}\exp(\mathrm{i}\mathbf{G}\cdot\mathbf{v}) = \Lambda\exp(\mathrm{i}\boldsymbol{\eta}\cdot\boldsymbol{v}\ )\boldsymbol{\eta}\exp(-\mathrm{i}\boldsymbol{\eta}\cdot\boldsymbol{v}\ ) = \Lambda\boldsymbol{\eta}.\end{aligned} \tag{66}$$

One easily verifies that transformations (65) and (66) leave matrices $F_1$ and $F_2$ invariant. □

In accordance with the lemma there are many possibilities how to generalize equation (60) to the case with anomalous interaction. Indeed, for any Galilean Bhabha equation there are matrices $S_a$, $\eta_a$ and $\Lambda$ for which conditions (63) are satisfied. For example, we can choose $\Lambda = \beta_0$. In addition, for many cases there exist a hermitizing matrix $\eta = \Lambda$ satisfying (63), see, e.g., equations (11) and (46). For particular representations of the algebra $hg(1,3)$ there are also other solutions of equations (63).

Thus the Pauli term for a Galilean Bhabha equation can be chosen in the form (64) or, more generally, as a linear combination of both $F_1$ and $F_2$. As a result we obtain the following equation:

$$Q\Psi \equiv \left(\beta_\mu\pi^\mu + \beta_4 m + \lambda_1\frac{e}{m}\beta_0\boldsymbol{\eta}\cdot\mathbf{H} + \lambda_2\frac{e}{m}\beta_0\,(\mathbf{S}\cdot\mathbf{H} - \boldsymbol{\eta}\cdot\mathbf{E})\right)\Psi = 0, \tag{67}$$

where $\lambda_1$ and $\lambda_2$ are dimensionless coupling constants.

Since equation (67) is reduced to equation (60) for $\lambda_1 = \lambda_2 = 0$, equation (67) describes anomalous as well as minimal interaction.

In order to receive the physical content of this equation it is convenient to apply the transformation

$$\Psi \rightarrow \Psi' = W^{-1}\Psi, \qquad Q \rightarrow Q' = W^{\dagger}QW, \tag{68}$$

where

$$W = \exp\left(-\mathrm{i}\frac{\boldsymbol{\eta}\cdot\pi}{m}\right) \tag{69}$$

and $\boldsymbol{\eta}$ is a vector whose components are the Galilei boost generators (9). For the case $e = 0$ (or $A_\mu = 0$) the operator $W$ reduces to operator $U$ given in equation (29), which was used for our analysis of free particle equations.

Using relations (7) and supposing that the nilpotence index $N$ of matrix $\boldsymbol{\eta}\cdot\pi$ satisfies $N < 4$ we obtain the following equation:

$$\begin{aligned}Q'\Psi' \equiv \Bigg\{&\beta_0\left(\pi^0 - \frac{\boldsymbol{\pi}^2}{2m} + \frac{e}{m}\boldsymbol{\eta}\cdot\mathbf{F}\right) - \frac{e}{2m}\boldsymbol{\beta}\times\boldsymbol{\eta}\cdot\mathbf{H} + \beta_4 m - \frac{e}{6m^2}\widehat{Q}_{ab}\frac{\partial H_a}{\partial x_b}\\ &+\frac{e}{m}\Lambda\left[\lambda_1\boldsymbol{\eta}\cdot\mathbf{H} + \lambda_2\left(\mathbf{S}\cdot\mathbf{H} - \boldsymbol{\eta}\cdot\mathbf{F} + \frac{1}{2m}\tilde{Q}_{ab}\frac{\partial H_a}{\partial x_b}\right)\right]\Bigg\}\Psi' = 0\end{aligned} \tag{70}$$

which is equivalent to (67). Here $\mathbf{E} = -\nabla A^0 - \frac{\partial\mathbf{A}}{\partial t}$ and $\mathbf{H} = \nabla\times\mathbf{A}$ are vectors of the corresponding electric and magnetic field strength, respectively,

$$\mathbf{F} = \mathbf{E} + \frac{1}{2m}(\boldsymbol{\pi}\times\mathbf{H} - \mathbf{H}\times\boldsymbol{\pi}) \tag{71}$$

and

$$\tilde{Q}_{ab} = \tfrac{1}{2}(\eta_a S_b + \eta_b S_a + S_b\eta_a + \eta_a S_b), \qquad \widehat{Q}_{ab} = \eta_a^{\dagger}\varepsilon_{bcd}\beta_c\eta_d + \varepsilon_{bcd}\beta_c\eta_d\eta_a.$$

Equation (70) includes the Schrödinger terms $\left(\pi^0 - \frac{\pi^2}{2m}\right)\Psi'$ and additional terms which are linear in vectors of the external field strengths and their derivatives.

Note that if the nilpotence index $N$ of matrices $\eta_a$ satisfies the relation $N < 4$, which is fulfilled for all representations of algebra $hg(1,3)$ considered in the present paper, the transformed equation (70) is completely equivalent to initial equation (67).

*4.3. The Galilean equation for spinor particle interacting with an external field*

Let us consider equation (70) for two particular realizations of $\beta$-matrices in more detail. First note, that our conclusions from equations (60)–(70) are true in general and in particular for the Lévy-Leblond equation, i.e., when $\beta_\mu, \beta_4$ are $4\times 4$ matrices determined by relations (10) with $\omega = \kappa = 0$. Then $\beta_0\eta_a = 0$, $\widehat{Q}_{ab} = \tilde{Q}_{ab} = 0$, $\boldsymbol{\beta}\times\boldsymbol{\eta} = -2\beta_0\mathbf{S}$, and equation (70) is reduced to the following form:

$$\left\{\beta_0\left(\pi^0 - \frac{\boldsymbol{\pi}^2}{2m} + \frac{e}{m}\mathbf{S}\cdot\mathbf{H}\right) + \beta_4 m + \frac{e}{m}\Lambda\left[\lambda_1\boldsymbol{\eta}\cdot\mathbf{H} + \lambda_2\left(\mathbf{S}\cdot\mathbf{H} - \boldsymbol{\eta}\cdot\mathbf{F}\right)\right]\right\}\Psi' = 0. \tag{72}$$

For $\lambda_1 = \lambda_2 = 0$ (i.e., when only the minimal interaction is present) equation (72) is reduced to the following system:

$$\left(\pi_0 - \frac{\boldsymbol{\pi}^2}{2m} + \frac{e}{2m}\boldsymbol{\sigma}\cdot\mathbf{H}\right)\varphi_1 = 0 \tag{73}$$

and

$$m\varphi_2 = 0, \qquad \texttt{or} \qquad \varphi_2 = 0, \tag{74}$$

where $\varphi_1 = \beta_0\Psi'$ and $\varphi_2 = (1-\beta_0)\Psi'$ are two-component spinors.

Thus introducing the minimal interaction (57) into the Lévy-Leblond equation, we get the Pauli equation for physical components of the wavefunction; moreover, the coupling constant (gyromagnetic ratio) for the Pauli interaction $\frac{e}{2m}\hat{\mathbf{s}}\cdot\mathbf{H}$, $\hat{\mathbf{s}} = \frac{1}{2}\boldsymbol{\sigma}$ has the same value $g = 2$ as in the case of the Dirac equation [3].

Considering now the case with an anomalous interaction we conclude that the general form of matrix $\Lambda$ satisfying relations (63) is

$$\Lambda = \nu\beta_0 + \mu\eta, \tag{75}$$

where $\eta$ is the hermitizing matrix (11), $\nu$ and $\mu$ are arbitrary parameters. Substituting (75) into (72) we obtain the following generalization of system (73):

$$\left(\pi_0 - \frac{\boldsymbol{\pi}^2}{2m} + \frac{eg}{2m}\boldsymbol{\sigma}\cdot\mathbf{H} - \frac{e\lambda_3}{2m}\boldsymbol{\sigma}\cdot\mathbf{F} - \frac{\lambda_3^2 e^2}{8m^3}\mathbf{H}^2\right)\varphi_1 = 0, \qquad \varphi_2 = -\frac{\lambda_3 e}{4m}\boldsymbol{\sigma}\cdot\mathbf{H}\varphi_1, \tag{76}$$

where $g = 2 + \mu\lambda_1 + \nu\lambda_2$ and $\lambda_3 = \mu\lambda_2$ are arbitrary parameters.

We see that the Lévy-Leblond equation with minimal and anomalous interactions reduces to the Galilean Schrödinger–Pauli equation (76) which, however, includes two additional terms linear in strength of an electric field and linear and quadratic in strength of a magnetic field. We shall discuss them in detail in section 4.6.

*4.4. The Galilean Duffin–Kemmer equation for particle interacting with an external field*

Consider now the Galilean Duffin–Kemmer equation for spin one particles interacting with an external field. The corresponding $\beta$-matrices in (67) have dimension $10\times 10$ and are given explicitly in (44). Thus

$$\beta_0\boldsymbol{\eta} = \begin{pmatrix} \mathbf{s} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1}\\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1}\\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1}\\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & 0 \end{pmatrix}, \qquad \beta_0\mathbf{S} = \begin{pmatrix} \mathbf{0}_{3\times3} & \mathbf{s} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1}\\ \mathbf{s} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1}\\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1}\\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & 0 \end{pmatrix},$$

$$\widehat{Q}_{ab} = \begin{pmatrix} -3Q_{ab} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1}\\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1}\\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1}\\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & 0 \end{pmatrix}, \qquad \tilde{Q}_{ab} = \begin{pmatrix} Q'_{ab} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1}\\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1}\\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1}\\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & 0 \end{pmatrix},$$

$$\beta \times \eta \cdot \mathbf{H} = -\begin{pmatrix} \mathbf{0}_{3\times3} & \mathbf{s}\cdot\mathbf{H} & \mathbf{0}_{3\times3} & 2\mathbf{k}^\dagger\cdot\mathbf{H} \\ \mathbf{s}\cdot\mathbf{H} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1} \\ 2\mathbf{k}\cdot\mathbf{H} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & 0 \end{pmatrix}, \tag{77}$$

where

$$Q_{ab} = s_a s_b + s_b s_a - \tfrac{4}{3}\delta_{ab} \qquad \text{and} \qquad Q'_{ab} = Q_{ab} + \tfrac{4}{3}\delta_{ab}. \tag{78}$$

Let us consider the corresponding equation (70) and restrict ourselves to $\Lambda = \beta_0$. Representing $\Psi'$ as a column vector $(\psi_1, \psi_2, \psi_3, \varphi)$, where $\psi_1, \psi_2, \psi_3$ are three-component vector functions and $\varphi$ is a one-component scalar function and using (44), (77), (78) we reduce (70) to the following Pauli-type equation for $\psi_1$:

$$\mathrm{i}\frac{\partial}{\partial t}\psi_1 = \widehat{H}\psi_1, \tag{79}$$

where

$$\widehat{H} = \frac{\nu^2}{2}m + \frac{\boldsymbol{\pi}^2}{2m} + eA_0 - \frac{ge}{2m}\mathbf{s}\cdot\mathbf{H} + \frac{qe}{\nu m}\mathbf{s}\cdot\mathbf{E} - \frac{qe}{2\nu m^2}\mathbf{s}\cdot(\boldsymbol{\pi}\times\mathbf{H} - \mathbf{H}\times\boldsymbol{\pi})$$
$$+ \frac{e}{2\nu m^2}(2-q)Q_{ab}\frac{\partial H_a}{\partial x_b} + \frac{e^2}{2\nu^2 m^3}(\mathbf{H}^2 - (\mathbf{s}\cdot\mathbf{H})^2) \tag{80}$$

and $g = 1 + 2\lambda_1 + 2\lambda_2, \quad q = 1 - \lambda_2$.

The remaining components of $\Psi'$ can be expressed in terms of $\psi_1$

$$\varphi = -\frac{e}{\nu^2 m^2}\mathbf{k}\cdot\mathbf{H}\psi_1, \qquad \psi_2 = -\frac{1}{\nu}\psi_1,$$
$$\psi_3 = -\nu\psi_2 - \frac{1}{m}\left(\pi_0 - \frac{1}{2m}\boldsymbol{\pi}^2 + \frac{e}{2m}\mathbf{s}\cdot\mathbf{H}\right)\psi_1.$$

Note, that in comparison with (76) equation (79) has an essentially new feature. Namely, excluding anomalous interaction, i.e., setting $\lambda_1 = \lambda_2 = 0$ in (80) it still includes the term $-\frac{e}{\nu m}\mathbf{s}\cdot\mathbf{E}$ describing the coupling of spin with an electric field. We shall show in the following section that this effectively represents the spin–orbit coupling. The other terms of Hamiltonian (80) (which are placed in the second line of equation (80)) can be neglected starting with a reasonable assumption about possible values of the magnetic field strength.

However, equation (76) for $\lambda_1 = \lambda_2 = 0$ reduces to the Schrödinger–Pauli equation (73) which has nothing to do with the spin–orbit coupling.

### *4.5. The Galilean Proca equation for interacting particles*

Equations (48) and (49) can be generalized too by introducing minimal and anomalous interactions with an external field. As a result we obtain

$$\pi^{\mathrm{k}}\Psi^{\mathrm{n}} - \pi^{\mathrm{n}}\Psi^{\mathrm{k}} = m\Psi^{\mathrm{kn}},$$
$$\pi_{\mathrm{k}}\Psi^{\mathrm{nk}} = \nu m\Psi^{\mathrm{n}} + \mathrm{i}\mu\frac{e}{m}F^{\mathrm{nk}}\Psi_{\mathrm{k}} + \lambda\delta^{\mathrm{n}4}m\Psi^4, \tag{81}$$

where parameters $\lambda$ and $\nu$ satisfy the conditions $\lambda\nu = 0, \nu^2 + \lambda^2 \neq 0$. Formulae (81) generalize both equation (48) (which corresponds to $\nu = 0, \lambda \neq 0$) and equation (49) (for which $\nu \neq 0, \lambda = 0$).

Multiplying equations (81) by both $\pi_{\mathrm{n}}$ and $\pi_{\mathrm{n}}\pi_{\mathrm{k}}$, summing up by $m$ and $k$ and then expressing $\Psi^{\mathrm{nk}}$, $\Psi^0$ via $\Psi^a$ and $\Psi^4$, we obtain the following system:

$$\begin{aligned}&\left(\pi_0-\frac{\pi^2}{2m}+\frac{1+2\mu}{2m}\mathbf{s}\cdot\mathbf{H}-\frac{\nu}{2}m\right)\Psi+\frac{e}{2m}(1-\mu)\mathbf{k}^\dagger\cdot\mathbf{F}\Psi^4=0,\\&\left(\nu\pi_0+\frac{(\lambda-\nu^2)}{2m}\right)\Psi^4-\frac{e}{2m}(1-\mu)\mathbf{k}\cdot\mathbf{F}\Psi=0,\end{aligned}\tag{82}$$

where we have denoted $\Psi=\mathtt{column}(\Psi^1,\Psi^2,\Psi^3)$.

Let $\nu=0$ then solving the second of equations (82) for $\Psi^4$ and substituting the result into the first equation we obtain

$$\left(\pi_0-\frac{\pi^2}{2m}+(1+2\mu)\frac{e}{2m}\mathbf{s}\cdot\mathbf{H}-\frac{\nu}{2}m+(1-\mu)^2\frac{e^2}{2m^3}(\mathbf{F}^2-(\mathbf{s}\cdot\mathbf{F})^2\right)\Psi=0.\tag{83}$$

In accordance with (83) the Galilean Proca equation for a particle with spin 1 interacting with an external field is reduced to the Schrödinger–Pauli equation with an extra term $\frac{e^2}{2m^3}(\mathbf{F}^2-(\mathbf{s}\cdot\mathbf{F})^2)$ which can be treated as a small correction.

For $\nu\neq0$ equations (82) describe a composed system with spins $s=1$ and $s=0$. Note that there are two privileged values of the arbitrary parameter $\nu$, namely, $\nu=-1$ and $\nu=1$.

If $\nu=-1$ or $\nu=1$ then equations (82) can be rewritten in the Schrödinger form (79), where $\psi_1=\mathtt{column}(\Psi^1,\Psi^2,\Psi^3,\Psi^4)$, and

$$\hat{H}=\frac{\pi^2}{2m}+eA_0-\frac{ge}{2m}\mathbf{S}\cdot\mathbf{H}-\frac{3-g}{2m}\mathbf{K}\cdot\mathbf{F}-\frac{m}{2}\tag{84}$$

or

$$\hat{H}=\frac{\pi^2}{2m}+eA_0-\frac{ge}{2m}\mathbf{S}\cdot\mathbf{H}-\frac{3-g}{2m}\hat{\mathbf{K}}\cdot\mathbf{F}+\frac{m}{2},\tag{85}$$

respectively. Here $g=1+2\mu$ and $\mathbf{S}$, $\mathbf{K}\hat{\mathbf{K}}$ are matrix vectors whose components are

$$S_a=\begin{pmatrix}s_a&\mathbf{0}_{3\times1}\\\mathbf{0}_{1\times3}&0\end{pmatrix},\qquad K_a=\begin{pmatrix}\mathbf{0}_{3\times3}&k_a^\dagger\\k_a&0\end{pmatrix},\qquad \hat{K}_a=\begin{pmatrix}\mathbf{0}_{3\times3}&k_a^\dagger\\-k_a&0\end{pmatrix}\tag{86}$$

and $k_a$ are matrices (16). Matrices $\{S_a,K_a\}$ and $\{S_a,\hat{K}_a\}$ form bases of the algebra $so(4)$ and $so(1,3)$, respectively.

Note that Hamiltonian (84) is formally Hermitian w.r.t. the standard scalar product for the direct sum of four square integrable functions while (85) is Hermitian w.r.t. the indefinite metric $(\psi_1,\psi_2)=\int\psi_1^\dagger M\psi_2\mathrm{d}^3x$, where $M$ is either a matrix whose elements are given in equation (50) or a parity operator. For other non-vanishing values of the parameter $\nu$ in (81), i.e., for $\nu\neq0,\pm1$, the corresponding Hamiltonian $\hat{H}$ is non-Hermitian.

### *4.6. The Galilei invariance and spin–orbit coupling*

Consider now the first of equations (76) for particular values of arbitrary parameters, namely for $\tilde{\lambda}_1+\tilde{\lambda}_2=-1$

$$\hat{L}\varphi_1\equiv\left(\pi_0-\frac{\boldsymbol{\pi}^2}{2m}-\frac{e\lambda_3}{2m}\boldsymbol{\sigma}\cdot\mathbf{F}-\frac{\lambda_3^2e^2}{8m^3}\mathbf{H}^2\right)\varphi_1=0.\tag{87}$$

First, let us recall that this equation is a direct consequence of the Galilei-invariant Lévy-Leblond equation with an anomalous interaction, i.e., of equation (67) where $\beta_{\mathrm{n}}$ are matrices (10) with $\kappa=\omega=0$. Second, equation (87) by itself is transparently Galilei-invariant since the operator $\hat{L}$ in ( 87) is a Galilean scalar provided the value of an arbitrary parameter $\lambda_3$ is finite. We shall assume $\lambda_3$ to be small.

In order to find the physical content of equation (87) we transform it to a more transparent form using the operator $U = \exp(-\frac{\mathrm{i}\lambda_3}{2m}\boldsymbol{\sigma}\cdot\pi)$. Applying this operator to $\varphi_1$ and transforming $\hat{L} \to \hat{L}' = U\hat{L}U^{-1}$ we obtain the equation

$$L'\varphi_1' = \left(\pi_0 - \frac{\boldsymbol{\pi}^2}{2m} - eA_0 - \frac{e\lambda_3^2}{8m^2}(\boldsymbol{\sigma}\cdot(\boldsymbol{\pi}\times\mathbf{E} - \mathbf{E}\times\boldsymbol{\pi}) - \mathtt{div}\mathbf{E}) + \cdots\right)\varphi_1' = 0, \qquad (88)$$

where the dots denote small terms of the order $o(\lambda_3^3)$ and $o(e^2)$.

All terms in big round brackets have an exact physical meaning. They include first the Schrödinger terms $\pi_0 - \frac{\pi^2}{2m} - eA_0$, then the term $\sim \boldsymbol{s}\cdot(\boldsymbol{\pi}\times\mathbf{E} - \mathbf{E}\times\boldsymbol{\pi})$ describing a spin–orbit coupling and, finally, a term $\sim\mathtt{div}\ \mathbf{E}$, i.e., a Darwin coupling.

Similarly, starting with equation (79), setting $\lambda_2 = -1, \lambda_1 = \frac{1}{2}$, supposing $\frac{1}{\nu}$ to be a small parameter and making use of the transformation $\psi_1 \to \psi_1' = \hat{U}\psi_1$ with $\hat{U} = \exp(-\frac{2\mathrm{i}}{\nu m}\mathbf{s}\cdot\boldsymbol{\pi})$ we obtain the equation

$$\left(\pi_0 - \frac{\nu^2}{2}m - \frac{\boldsymbol{\pi}^2}{2m} - eA_0 - \frac{\lambda e}{m^2}\left(\boldsymbol{s}\cdot(\boldsymbol{\pi}\times\mathbf{E} - \mathbf{E}\times\boldsymbol{\pi}) + \frac{4}{3}\mathtt{div}\mathbf{E} - Q_{ab}\frac{\partial E_a}{\partial x_b}\right) + \cdots\right)\psi_1' = 0. \qquad (89)$$

Here, $\lambda = \frac{2}{\nu^2}$ and the dots denote small terms of the order $o(\frac{1}{\nu^3})$ and $o(e^2)$.

Like equation (88), equation (89) includes the terms which describe the spin–orbit and Darwin couplings. Let us stress that these terms are kept for the minimal coupling also, i.e., when we set $\lambda_1 = \lambda_2 = 0$ in (67) and (80). In addition, there is the term $\sim Q_{ab}\frac{\partial E_a}{\partial x_b}$ which describes a quadrupole interaction of a charged vector particle with an electric field.

Analogously we can analyse equation (79) with the Hamiltonians (84) and (85). Setting there $g = 0$ and making transformations $\hat{H} \to U\hat{H}U^{-1} - \mathrm{i}U\frac{\partial U^{-1}}{\partial t}$, where $U = \exp(-3\mathrm{i}\mathbf{K}\cdot\boldsymbol{\pi}/2m)$ and $U = \exp(-3\mathrm{i}\hat{\mathbf{K}}\cdot\boldsymbol{\pi}/2m)$ for Hamiltonians (84) and (85), respectively, we obtain the approximate equation (89) with $\lambda = 9\nu/8$, $\nu = \mp 1$.

Thus we again come to the conclusion (see [21]) that the spin–orbit and Darwin couplings can be effectively described within the framework of a Galilei-invariant approach and thus they have not to be necessarily interpreted as purely relativistic effects.

Let us note that it is possible to choose parameters $\lambda_1$ and $\lambda_2$ in (80) in such a way that the anomalous interaction with an electric field will not be present. Namely, we can set $\lambda_2 = 1, \lambda_1 = -1/2$, and obtain, instead of (89), the following equation:

$$\left(\pi_0 - \frac{\nu^2}{2}m - \frac{\boldsymbol{\pi}^2}{2m} - eA_0 + \frac{ge}{2m}\mathbf{s}\cdot\mathbf{H} + \cdots\right)\psi_1' = 0, \qquad (90)$$

where $g = 2$. In other words, introducing a specific anomalous interaction into the Galilean Duffin–Kemmer equation we can reduce it to the Schrödinger–Pauli equation with the correct value of gyromagnetic ratio $g$.

## 5. Discussion

It is pretty well known that the correct definition of non-relativistic limit of relativistic theories is by no means a simple problem. In particular, as it was noted once more in the recent paper [32], a straightforward non-relativistic expansion in terms of $v/c$ leads to losing either Galilei invariance or important contributions such as spin–orbit coupling. Thus to make this limit correctly it is imperative to have *a priori* information on possible Galilean limits of a given theory.

In the present paper we continue the study of the Galilei-invariant theories for vector and spinor fields, started in [5–7]. The peculiarity of our approach is that, as distinct to the other approaches (e.g., to [18–23, 28] ), it enables to find out *a complete list of the Galilei-invariant equations for massive scalar and vector fields*. This possibility is due to our knowledge of all non-equivalent indecomposable representations of the Galilei algebra $hg(1,3)$ that can be constructed on representation spaces of scalar and vector fields, i.e., the representations which were described for the first time in paper [5]. Note that Galilean equations for massless vector and scalar fields have been presented in our previous paper [7].

Using this complete list of representations we find all systems of the first-order Galilei-invariant wave equations (1) for scalar and vector fields. The $\beta$-matrices for these Galilei-invariant wave equations are given in appendix A. In fact we have described how to construct any wave equation of finite order invariant with respect to the Galilei group since such an equation can be written as a first-order differential equation in which various derivatives of fields are considered as new variables of $\Psi$.

Then Galilean analogues of some popular relativistic equations for vector particles and particles with spin 3/2 are described, in particular, the Galilean Proca and the Galilean Rarita–Schwinger equations. However these Galilean equations are not non-relativistic limits of the relativistic Proca or relativistic Rarita–Schwinger equations since, among other things, they have more components. Thanks to that it is possible to obtain equations which keep all the main features of their relativistic analogues. To the best of our knowledge this is done for the first time in the present paper.

We pay special attention to the description of the Galilean particles interacting with an external electromagnetic field. We study both the cases with a minimal interaction as well as anomalous one.

A quite general form of an anomalous interaction which satisfies the Galilei invariance condition is written in equation (67). It contains two coupling constants, $\lambda_1$ and $\lambda_2$, whose values can be fixed via physical reasoning. If we fix the value of the gyromagnetic ratio predicted by (67) then the randomness in the description of anomalous interaction is reduced to one arbitrary parameter. It can be fixed too if we restrict ourselves to a desired value of the spin–orbit coupling constant.

Note that the results presented in sections 4.1 and 4.3 are valid for arbitrary equations (67) invariant with respect to the Galilei group whereas those in sections 4.2, 4.4 and 4.5 are true for special equations with anomalous interactions, i.e., for the Galilean Proca equation, the generalized Lévy-Leblond equation and for a generalized Galilean Duffin–Kemmer equation. We shall show that the last equations describe consistently charged particles interacting with an electromagnetic field. In other words, they describe an important physical effect, namely, spin–orbit coupling which is, however, traditionally interpreted as a purely relativistic phenomenon.

On the other hand let us note that there are some principal difficulties with the Galilean approach since the Galilei invariance requires that mass and energy are separately conserved, and that within the Galilean theories there is no concept of proper time which yields a phase effect that does not depend on the velocity of light and so does not disappear in a non-relativistic limit. Of course, there are obvious restrictions to phenomena which are characterized by velocities much smaller than that of light. Moreover, there are also problems in our approach with the interpretation of undesired terms $\sim Q_{ab}\frac{\partial H_a}{\partial x_b}$ and $\mathbf{s}\cdot(\boldsymbol{\pi}\times\mathbf{H}-\mathbf{H}\times\boldsymbol{\pi})$ which appear in Hamiltonian (79). Thanks to an appropriate choice of otherwise arbitrary parameters $\lambda_1$ and $\lambda_2$; these terms are not present in effective Hamiltonians (89) as well as (90) which describe spin–orbit and the Pauli couplings, respectively. However, if we would like to keep both these couplings, then the undesired terms may appear.

Another problem is connected with a sign in front of the terms describing the spin–orbit and the Darwin couplings. Comparing (88) with the quasi-relativistic approximation of the

Dirac equation we conclude, that in order to obtain a correct sign it is necessary to suppose that $\lambda_3$ be pure imaginary. Moreover, for $\lambda_3 = \mathrm{i}$ the coupling constants for spin–orbit and the Darwin interactions in (88) coincide with the relativistic ones predicted by the Dirac equation.

Note that the exact equation (87) is much simpler then the approximate equation (88) and can be solved exactly for some particular external fields (for instance, the Coulomb ones). However, if $\lambda_3$ is imaginary, the term $-\frac{e\lambda_3}{2m}\boldsymbol{\sigma}\cdot\mathbf{F}$ in equation (87) and the corresponding Hamiltonian $\hat{H} = -L + p_0 = A_0$ are non-Hermitian. If $A_0$ and $\mathbf{A}$ are even and odd functions of $\mathbf{x}$, respectively, equation (87) appears to be invariant with respect to a product of the space inversion $P$ and the Wigner time inversion $T$ (compare with [33]), and thus serves as an example of a $PT$-symmetric quantum-mechanical system.

Any Galilean theory, by definition, is only an approximation of the corresponding relativistic one. Thus the very existence of a physically consistent non-relativistic approximation can serve as a consistency criterion of a relativistic theory. Thus our study of the Galilean wave equations has contributed to the theory of relativistic equations, since effectively we have analysed possible non-relativistic limits of theories for vector and scalar particles. On the other hand the results of the present paper can be applied to purely non-relativistic models satisfying the Galilei invariance criteria or being invariant w.r.t. various extensions of the Galilei group, e.g., w.r.t. Galilei supergroup [34].

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## Appendix A. Submatrices *R* and *E* of matrices $\beta_4$

Here we present all non-trivial solutions of equations (21) which give rise to explicit forms of matrices $\beta_4$ given by equation (20). The associated matrices $\beta_0$ and $\beta_a$ are given by equations (22) and (23).

Solving equations (21), where $A, C$ and $A', C'$ are matrices given in table 1 which correspond to $q = (n, m, \lambda)$ and $q' = (n', m', \lambda')$, respectively, we obtain the associated matrices $R = R(q, q'), E = E(q, q')$ (which determine matrix $\beta_4$ via (20)) in the forms presented in tables A1–A3, where the Greek letters denote arbitrary real parameters. These matrices are simplified using the equivalence transformations

$$R \to URU^\dagger, \qquad E \to VEV^\dagger. \tag{A.1}$$

Here $U$ and $V$ are unitary matrices whose dimensions are the same as dimensions of matrices $R$ and $E$ correspondingly, which satisfy the following relations: $UA = AU, UB = BV, VC = CU$. Transformations (A.1) keep equations (21) invariant.

## Appendix B. More on the Galilean Rarita–Schwinger equation

Let us prove that equation (55) is consistent and describes a particle with spin $s = 3/2$.

Reducing (55) by $p_{\mathtt{m}}$ we obtain $\lambda m^2\hat{\Psi}^4 = 0$, i.e., $\hat{\Psi}^4 = 0$. Whereas, reducing (55) by $\hat{\gamma}_{\mathtt{m}}$ we obtain

$$p_{\mathtt{n}}\hat{\Psi}^{\mathtt{n}} = \hat{\gamma}_{\mathtt{n}}p^{\mathtt{n}}\hat{\gamma}_{\mathtt{m}}\hat{\Psi}^{\mathtt{m}}. \tag{B.1}$$

**Table A1.** Submatrices $R$ and $E$ of matrices $\beta_4$.

| | $m, n, \lambda$ | | |
|---|---|---|---|
| $m', n', \lambda'$ | 3, 1, 1 | 2, 2, 1 | 2, 1, 0 |
| 3, 1, 1 | $R = \begin{pmatrix} \mu & \nu & \sigma \\ \nu & \alpha & \lambda \\ \sigma & \lambda & 0 \end{pmatrix}$ $E = \alpha - 2\sigma$, $\mu\nu = 0, \lambda(\alpha - \sigma) = 0$ | $R = \begin{pmatrix} \mu & \sigma & \omega \\ \nu & \alpha & 0 \end{pmatrix}$ $E = \begin{pmatrix} \kappa \\ \omega - \alpha \end{pmatrix}$ | $R = \begin{pmatrix} \mu & \sigma & 0 \\ \nu & \alpha & \omega \end{pmatrix}$ $E = \kappa$ |
| 2, 2, 1 | $R = \begin{pmatrix} \mu & \nu \\ \sigma & \alpha \\ \omega & 0 \end{pmatrix}$ $E = (\kappa(\omega - \alpha))$ | $R = \begin{pmatrix} \mu & \nu \\ \nu & \kappa \end{pmatrix}, \mu\nu = 0$ $E = \begin{pmatrix} \sigma & 0 \\ 0 & \omega \end{pmatrix}$ | $R = \begin{pmatrix} \mu & \sigma \\ \nu & \omega \end{pmatrix}$ $E = (\kappa\omega)$ |
| 2, 1, 0 | $R = \begin{pmatrix} \mu & \nu \\ \sigma & \alpha \\ 0 & \omega \end{pmatrix}$ $E = \kappa$ | $R = \begin{pmatrix} \mu & \nu \\ \sigma & \omega \end{pmatrix}, E = \begin{pmatrix} \kappa \\ \omega \end{pmatrix}$ | $R = \begin{pmatrix} \mu & \nu \\ \nu & \kappa \end{pmatrix}$ $E = \sigma, \mu\nu = 0$ |
| 2, 1, 1 | $R = \begin{pmatrix} \mu & \nu \\ \sigma & \alpha \\ \omega & 0 \end{pmatrix}$ $E = \omega - \alpha$ | $R = \begin{pmatrix} \mu & \nu \\ 0 & \omega \end{pmatrix}, E = \begin{pmatrix} \alpha \\ \sigma \end{pmatrix}$ | $R = \begin{pmatrix} \mu & \sigma \\ 0 & \nu \end{pmatrix}$ $E = \kappa$ |
| 2, 0, 0 | $R = \begin{pmatrix} \mu & \nu \\ \sigma & \alpha \\ \alpha & 0 \end{pmatrix}$ $E$ not existing | $R = \begin{pmatrix} \mu & \nu \\ \omega & 0 \end{pmatrix}$ $E$ not existing | $R = \begin{pmatrix} \mu & \nu \\ \sigma & 0 \end{pmatrix}$ $E$ not existing |
| 1, 2, 1 | $R = \begin{pmatrix} \mu \\ \nu \\ \alpha \end{pmatrix}$ $E = (\omega\alpha)$ | $R = \begin{pmatrix} \kappa \\ \sigma \end{pmatrix}, E = \begin{pmatrix} \mu & \nu \\ \omega & 0 \end{pmatrix}$ | $R = \begin{pmatrix} \mu \\ \nu \end{pmatrix}$ $E = (\sigma 0)$ |
| 1, 1, 0 | $R = \begin{pmatrix} \mu \\ \nu \\ \alpha \end{pmatrix}, E = \alpha$ | $R = \begin{pmatrix} \kappa \\ \sigma \end{pmatrix}, E = \begin{pmatrix} \mu \\ 0 \end{pmatrix}$ | $R = \begin{pmatrix} \mu \\ \nu \end{pmatrix}$ $E = \sigma$ |
| 1, 1, 1 | $R = \begin{pmatrix} 0 \\ \nu \\ \alpha \end{pmatrix}, E = \omega$ | $R = \begin{pmatrix} \kappa \\ \sigma \end{pmatrix}, E = \begin{pmatrix} \mu \\ \nu \end{pmatrix}$ | $R = \begin{pmatrix} \mu \\ \nu \end{pmatrix}$ $E = 0$ |
| 1, 0, 0 | $R = \begin{pmatrix} \mu \\ \alpha \\ 0 \end{pmatrix}$ $E$ not existing | $R = \begin{pmatrix} \kappa \\ \sigma \end{pmatrix}$ $E$ not existing | $R = \begin{pmatrix} \kappa \\ \sigma \end{pmatrix}$ $E$ not existing |
| 0, 1, 0 | $R$ not existing, $E = \alpha$ | $R$ not existing, $E = \begin{pmatrix} \kappa \\ \sigma \end{pmatrix}$ | $E = \alpha$, $R$ not existing |

Finally, comparing (B.1) with equation (55) for index $\mathtt{m} = 4$ we find the following consequences of equation (55):

$$\hat{\gamma}_{\mathtt{n}} p^{\mathtt{n}} \hat{\Psi}^{\sigma} = 0, \qquad \sigma = 0, 1, 2, 3, \tag{B.2}$$

$$m\hat{\Psi}^0 - p^a \hat{\Psi}^a = 0, \qquad a = 1, 2, 3, \tag{B.3}$$

$$\hat{\gamma}_0 \hat{\Psi}^0 + \hat{\gamma}_a \hat{\Psi}^a = 0, \qquad \text{and} \qquad \hat{\Psi}^4 = 0. \tag{B.4}$$

On the other hand equation (55) follows from (B.2)–(B.4), so that equations (55) and (B.2)–(B.4) are equivalent.

In accordance with (B.2) any component of $\Psi^{\mathtt{m}}$ satisfies the Lévy-Leblond equation (compare with section 2.1). Let us prove now that equations (B.2)–(B.4) describe indeed a particle with spin $s = 3/2$.

**Table A2.** Submatrices $R$ and $E$ of matrices $\beta_4$.

| | $m, n, \lambda$ | | |
|---|---|---|---|
| $m', n', \lambda'$ | 2, 1, 1 | 2, 0, 0 | 1, 2, 1 |
| 2, 1, 1 | $R = \begin{pmatrix} \mu & \nu \\ \nu & 0 \end{pmatrix}$ $E = \sigma, \mu\nu = 0$ | $R = \begin{pmatrix} \omega & \nu \\ \mu & 0 \end{pmatrix}$ $E$ not existing | $R = (\mu\nu)$ $E = \begin{pmatrix} \sigma \\ \alpha \end{pmatrix}$ |
| 2, 0, 0 | $R = \begin{pmatrix} \omega & \mu \\ \nu & 0 \end{pmatrix}$ $E$ not existing | $R = \begin{pmatrix} \mu & \nu \\ \nu & 0 \end{pmatrix}$, $E$ not existing, $\mu\nu = 0$ | $R = (\mu \ \ \nu)$ $E$ not existing |
| 1, 2, 1 | $R = \begin{pmatrix} \mu \\ \nu \end{pmatrix}$ $E = (\sigma\alpha)$ | $R = \begin{pmatrix} \mu \\ \nu \end{pmatrix}$ $E$ not existing | $E = \begin{pmatrix} \mu & \nu \\ \nu & 0 \end{pmatrix}$, $\mu\nu = 0$; $R = \alpha$ |
| 1, 1, 0 | $R = \begin{pmatrix} \mu \\ \nu \end{pmatrix}$, $E = \sigma$ | $R = \mu$, $E$ not existing | $R = \mu$ $E = \begin{pmatrix} \nu \\ 0 \end{pmatrix}$ |
| 1, 1, 1 | $R = \begin{pmatrix} \mu \\ \nu \end{pmatrix}$, $E = \sigma$ | $R = \mu$, $E$ not existing | $R = \mu$, $E = \begin{pmatrix} \nu \\ \alpha \end{pmatrix}$ |
| 1, 0, 0 | $R = \begin{pmatrix} \kappa \\ \sigma \end{pmatrix}$ $E$ not existing | $R = \mu$, $E$ not existing | $R = \mu$, $E$ not existing |
| 0, 1, 0 | $R$ not existing, $E = \alpha$ | $R$ and $E$ not existing | $R$ not existing $E = \mu$ |

**Table A3.** Submatrices $R$ and $E$ of matrices $\beta_4$.

| | $m, n, \lambda$ | | | |
|---|---|---|---|---|
| $m', n', \lambda'$ | 1, 1, 0 | 1, 1, 1 | 1, 0, 0 | 0, 1, 0 |
| 1, 1, 0 | $R = \mu$ $E = \nu$ | $R = \mu$ $E = \nu$ | $R = \mu$, $E$ not existing | $E = \mu$, $R$ not existing |
| 1, 1, 1 | $R = \mu$, $E = \nu$ | $R = \mu$ $E = 0$ | $R = \mu$, $E$ not existing | $E = \mu$, $R$ not existing |
| 1, 0, 0 | $R = \mu$, $E$ not existing | $R = \mu$, $E$ not existing | $R = \mu$, $E$ not existing | $R$ and $E$ not existing |
| 0, 1, 0 | $E = \mu$, $R$ not existing | $E = \mu$, $R$ not existing | $R$ and $E$ not existing | $E = \mu$, $R$ not existing |

It follows from (B.2)–(B.4) that, in the rest frame, $\hat{\Psi}^0 = \hat{\Psi}^4 = 0$ and $\hat{\Psi}^a$ has only two nonzero spinor components $\hat{\Psi}^a_\alpha, \alpha = 1, 2$. Using $\hat{\gamma}$-matrices in realization (14) and equation (B.4), we conclude that $\hat{\Psi}^a$ satisfies the equation

$$\sigma_a \Psi^a = 0. \tag{B.5}$$

Consequently this function satisfies conditions (31) and (32) as well with $s = 3/2$. This follows from the fact the total spin operator $\mathbf{S}$ is a sum of operators of spin one and of spin one-half: $S_a = s_a + \frac{1}{2}\sigma_a$ , so that

$$\mathbf{S}^2 = \tfrac{11}{4} + \mathbf{s} \cdot \boldsymbol{\sigma}. \tag{B.6}$$

Let $\tilde{\Psi}$ denote the column $(\hat{\Psi}^1, \hat{\Psi}^2, \hat{\Psi}^3)$. In accordance with (B.6) the condition $\mathbf{S}^2\tilde{\Psi} = s(s+1)\tilde{\Psi}$ reduces to the form

$$\hat{\Psi}_a - \frac{\mathrm{i}}{2}\varepsilon_{abc}\sigma_b\hat{\Psi}_c = 0, \tag{B.7}$$

for $s = 3/2$ provided we use the representation with $(s_a)_{bc} = \mathrm{i}\varepsilon_{abc}$, where $\varepsilon_{abc}$ is a totally antisymmetric unit tensor.

Comparing (B.5) with (B.7) we conclude that these equations are completely equivalent since multiplying (B.5) by $\hat{\sigma}_a$ we obtain (B.7) and multiplying (B.7) by $\hat{s}_a$ and contracting it with respect to index $a$ we get (B.5).

Thus indeed equation (55) describes a Galilean particle with spin $s = 3/2$.

## Appendix C. Contraction of relativistic Proca equation

Let us consider the relativistic Proca equation (2) and contract it directly to a non-relativistic (i.e., Galilei-invariant) approximation. Solutions of this equation are a 4-vector $\Psi^\mu$ and a skew-symmetric tensor $\Psi^{\mu\nu}$ which transform according to the representation $D(\frac{1}{2},\frac{1}{2}) \oplus D(1,0) \oplus D(0,1)$ of the Lorentz group.

It was shown in papers [5] and [6] how this representation can be contracted to representation $D(3,1,1)$ of the homogeneous Galilei group. This contraction can be used to reduce equation (2) to a Galilei-invariant form. To do this it is necessary:

- To choose the following new dependent variables

$$R^a = -\tfrac{1}{2}(\Psi^{0a} + \Psi^a),\ N^a = \Psi^{0a} - \Psi^a,$$
$$W^c = \tfrac{1}{2}\varepsilon^{abc}\Psi_{bc},\ B = \Psi^0,$$

  which, in accordance with (2), satisfy the following equations:

$$\begin{aligned} &2(p^0 - \kappa)R^a + p^a B + \varepsilon^{abc} p_b W_c = 0, && (p^0 + \kappa)N^a - \varepsilon^{abc} p_b W_c + p^a B = 0, \\ &\varepsilon^{abc} p_b\big(R_c + \tfrac{1}{2}N_c\big) = \kappa W^a, && \tfrac{1}{2} p_a N^a - p_a R^a = \kappa B. \end{aligned} \tag{C.1}$$

- To act on variables $R^a$, $N^a$, $W^a$ and $B$ by a diagonal contraction matrix. This action yields the change

$$R^a = \tilde{R}^a, \qquad N^a = \varepsilon^2 \tilde{N}^a, \qquad W^a = \varepsilon \tilde{W}^a, \qquad B = \varepsilon \tilde{B},$$

  where $\varepsilon$ is a small parameter associated with the inverse speed of light.
- To change relativistic 4-momentum $p^\mu$ and mass $\kappa$ by their Galilean counterparts $\tilde{p}^a$, $\tilde{p}^0$ and $m$, where

$$\tilde{p}^a = \varepsilon^{-1} p^a, \qquad \tilde{p}^0 = p_0 - \kappa \qquad \text{and} \qquad m = \tfrac{1}{2}(p_0 + \kappa)\varepsilon^{-2}.$$

- Each equation in (C.1) keep only terms which are multiplied by the lowest powers of $\varepsilon$.

  As a result we obtain the following equations for $\tilde{R}^a$, $\tilde{N}^a$, $\tilde{W}^a$ and $\tilde{B}$:

$$2\tilde{p}^0\tilde{R}^a + \tilde{p}^a\tilde{B} + \varepsilon^{abc}\tilde{p}_b\tilde{W}_c = 0, \qquad \varepsilon^{abc}\tilde{p}_b\tilde{R}_c = m\tilde{W}^a, \qquad \tilde{p}_a\tilde{R}^a + m\tilde{B} = 0, \tag{C.2}$$

and

$$2m\tilde{N}^a = \varepsilon^{abc}\tilde{p}_b\tilde{W}_c - \tilde{p}^a\tilde{B}. \tag{C.3}$$

System (C.2) is nothing else but the Galilei-invariant equation (1) with matrices (42) written componentwise. Relation (C.3) expresses the extra component $\tilde{N}^a$ via derivatives of the essential ones, i.e. of $\tilde{W}_c$ and $\tilde{B}$.

Thus the Galilean analogue of the Proca equation (48) cannot be obtained as a non-relativistic limit of the relativistic Proca equation (2) but is a specific modification of it. The relativistic counterpart of equation (48) is a specific generalization of (2) which will be studied in a separate publication.

## References


[1] Wigner E P 1939 Unitary representations of Lorentz group *Ann. Math.* **40** 149–204

[2] Bargmann V 1954 On unitary ray representations of continuous groups *Ann. Math.* **59** 1–46

[3] Lévy-Leblond J M 1971 Galilei group and Galilean invariance *Group Theory and Applications* ed E M Loebl vol II (New York: Academic) pp 221–99

[4] Inönü E and Wigner E 1952 Representations of the Galilei group *Nuovo Cimento* B **9** 705–18

[5] de Montigny M, Niederle J and Nikitin A G 2006 Galilei invariant theories: I. Constructions of indecomposable finite-dimensional representations of the homogeneous Galilei group: directly and via contractions *J. Phys. A: Math. Gen.* **39** 1–21

[6] Niederle J and Nikitin A G 2006 Construction and classification of indecomposable finite-dimensional representations of the homogeneous Galilei group *Czech. J. Phys.* **56** 1243–50

[7] Niederle J and Nikitin A G 2009 Galilean equations for massless fields *J. Phys. A: Math. Gen.* **42** 105207

[8] Niederle J 1989 Relativistic wave equations based on indecompossible representations of $sl(2, C)$ *Symp. Math.* **31** 109–20

[9] Bhabha H J 1949 Relativistic wave equations for elementary particles *Rep. Math. Phys.* **17** 200–5
Gelfand I M and Yaglom I 1948 General relativistic wave equations and infinite-dimensional representations of the Lorentz group *J. Exp. Theor. Phys.* **18** 703–33

[10] Bruhat F 1956 Sur les representations induites des groupes de Lie *Bull. Soc. Math. France* **84** 97–205

[11] Gårding L 1944 *Medd. Lunds. Mat. Sem.* **6** 1

[12] Gel'fand I M, Minlos R A and Shapiro Z Ya 1963 *Representations of the Rotation and Lorentz Groups and their Applications* (New York: Pergamon)

[13] Corson E M 1955 *Introduction to Tensors, Spinors, and Relativistic Wave Equations* (London: Blackie & Sons)

[14] Proca A 1936 Fundamental equations of elementary particles *Compt. Rend.* **202** 1490

[15] Rarita W and Schwinger J 1941 On a theory of particles with half-integral spin *Phys. Rev.* **60** 61–1

[16] Singh L P H and Hagen C R 1974 Lagrangian formulation for arbitrary spin: 1. Boson case *Phys. Rev.* D **9** 898–909

[17] Niederle J and Nikitin A G 2001 Relativistic wave equations for interacting, massive particles with arbitrary half-integer spins *Phys. Rev.* D **64** 125013–24

[18] Lévy-Leblond J M 1967 Non-relativistic particles and wave equations *Commun. Math. Phys.* **6** 286–311

[19] Hagen C R and Hurley W J 1970 Magnetic moment of a particle with arbitrary spin *J. Phys. A: Math. Gen.* **24** 1381–4

[20] Hurley W J 1971 Nonrelativistic quantum mechanics for particles with arbitrary spin *Phys. Rev. D* **3** 2339–47

[21] Nikitin A G and Fuschich W I 1980 Equations of motion for particles of arbitrary spin invariant under the Galilei group *Theor. Math. Phys.* **44** 584–92

[22] Fushchich W I and Nikitin A G 1994 *Symmetries of Equations of Quantum Mechanics* (New York: Allerton Press)

[23] de Montigny M, Khanna F C, Santana A E and Santos E S 2001 Galilean covariance and the non-relativistic Bhabha equations *J. Phys. A: Math. Gen.* **34** 8901–17

[24] Kobayashi M, de Montigny M and Khanna F C 2007 Galilean covariant theories for Bargmann–Wigner fields with arbitrary spin *J. Phys. A: Math. Theor.* **40** 1117–40

[25] Kemmer N 1939 Quantum theory of Einstein–Bose particles and nuclear interaction *J. Phys. G: Nucl. Part. Phys.* **A 166** 127–53

[26] Fernandes M C B, Santana A E and Vianna J D M 2003 Galilean Duffin–Kemmer–Petiau algebra and symplectic structure *J. Phys. A: Math. Gen.* **36** 3841–54

[27] Le Bellac M and Lévy-Leblond J M 1973 Galilean electromagnetism *Nuovo Cimento* **14** 217–33

[28] Santos E S, de Montigny M, Khanna F C and Santana A E 2004 Galilean covariant Lagrangian models *J. Phys. A: Math. Gen.* **37** 9771–91

[29] Pauli W 1941 Relativistic field theories of elementary particles *Rev. Mod. Phys.* **13** 203–32

[30] de Montigny M, Khanna F C and Santana A E 2003 Nonrelativistic wave equations with gauge fields *Int. J. Theor. Phys.* **42** 649–71

[31] Foldy L L and Wouthuysen S A 1950 On the Dirac theory of spin 1/2 particles and its non-relativistic limit *Phys. Rev.* **78** 29–36

[32] Sulaksono A, Reinhard P-G, Buervenich T G, Hess P O and Maruhn J A 2007 From self-consistent covariant effective field theories to their Galilean-invariant counterparts *Phys. Rev. Lett.* **98** 252601

[33] Bender C M and Boettcher S 1998 Real spectra in non-Hermitian Hamiltonians having PT symmetry *Phys. Rev. Lett.* **80** 5243–6

[34] Henkel M and Unterberger J 2006 Supersymmetric extensions of Schrodinger-invariance *Nucl. Phys.* B **746** 155–201